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PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 35 March 15, 1949 Number 3

NATIONAL ACADEMY OF SCIENCES (DISTRIBUTION AND AGES OF MEMBERSHIP)

By Fred E. Wright, *Home Secretary*

Communicated January 29, 1949

The National Academy of Sciences was established by an Act of Congress, approved on March 3, 1863 by Abraham Lincoln. The number of charter members was 48; of these two declined membership. In the Act, the membership was limited to 50; the Academy was given power "to make its own organization including constitution and bylaws"; "to fill all vacancies created by death or resignation"; "to provide for the election of foreign and domestic members, the division into classes," "and to report the same to Congress." In addition, "the Academy shall, whenever called upon by any department of the Government, investigate, examine, experiment and report upon any subject of science or art, the actual expense of such investigations, examinations, experiments and reports to be paid from appropriations which may be made for the purpose, but the Academy shall receive no compensation whatever for any services to the Government of the United States."

The primary function of the Academy is therefore to serve as scientific adviser to the Government on problems in science and applied science for the solution of which a government department may seek unbiased, expert aid. Election to membership in the Academy means recognition of unusual ability; and it carries with it an obligation toward the Government. As a result, Academy membership is restricted largely to scientists who through research have contributed to increase in knowledge and are in position, upon request, to serve the Government in the solution of its problems in science.

The total Academy membership on July 1, 1948, numbered 435, including 5 members emeriti. Study of the membership with reference to distribution within the United States, to age and to fields of activity in science has been made at irregular intervals in the past. Early studies were reported upon by Raymond Pearl [Proc. Nat. Acad. Sci., **11**, 752–768 (1925)]. In 1933 and in 1935 F. E. Wright, as Home Secretary, continued the studies and circulated the results in mimeograph form to the memership of the Acad-

TABLE 1

DISTRIBUTION OF MEMBERSHIP BY STATES AND COUNTRIES (JULY 1, 1948)

	NO. BORN	NO. NOW
New England:		
Maine	5	2
Vermont	2	0
New Hampshire	4	0
Massachusetts	35	68
Connecticut	6	20
Rhode Island	1	2
Total	53	92
North Atlantic States:		
New York	42	75
New Jersey	4	28
Pennsylvania	24	16
Maryland	8	17
District of Columbia	2	29
Delaware	3	1
Total	83	166
South Atlantic States:		
Virginia	7	4
West Virginia	2	0
North Carolina	2	2
South Carolina	2	0
Georgia	4	0
Florida	0	2
Total	17	8
North Central States:		
Michigan	12	6
Wisconsin	20	13
Minnesota	5	4
North Dakota	1	0
South Dakota	4	0
Ohio	25	0
Indiana	18	4
Illinois	32	35
Iowa	12	4
Nebraska	5	0
Missouri	12	12
Kansas	10	0
Total	156	78

	NO. BORN	NO. NOW
South Central States:		
Kentucky	5	0
Tennessee	4	1
Arkansas	0	0
Oklahoma	2	0
Alabama	1	0
Mississippi	0	0
Louisiana	1	0
Texas	4	6
Total	17	7
Rocky Mountain States:		
Montana	0	0
Wyoming	1	0
Idaho	0	0
Colorado	5	1
Utah	3	1
Nevada	0	0
New Mexico	1	1
Arizona	0	1
Total	10	4
Pacific States:		
Washington	3	0
Oregon	1	0
California	17	80
Total	21	80

Foreign Born:			
Austria	4	Italy	1
Canada	11	Jugoslavia	1
Denmark	2	Mexico	1*
England	5	Norway	2
Estonia	1	Poland	3
France	1	Russia	9
Germany	11	Scotland	3
Holland	4	Sweden	6
Hungary	4	Switzerland	4
India	2*	Syria	1*
Ireland	1	Ukraine	1

* American parents.

emy. In 1945 Edwin B. Wilson published a short report on "Vital Statistics of the National Academy of Sciences" [PROC. NAT. ACAD. SCI., **31**,

200–202 (1945)] and stressed the slowly rising average age of members at election.

Request has been made recently for a revision of the 1935 statement, based on the present membership of the Academy. Since 1935 the number of Academy members has increased from 289 to 435; this rapid growth has been chiefly due to action by the Academy in 1942 which raised the limit from 15 to 30 in the number of men who may be elected at any annual meeting. The result has been that more than one-half the Academy members, now living, were elected during the ten-year period 1939 to 1948, inclusive.

Regional Distribution of Members.—In table 1 data on the distribution of the membership in the several states are given. The table shows that in New England, the North Atlantic States and California the number of Academy resident members greatly exceeds the number of Academy members born therein; the ratio is 337 to 157. In the North Central States, 156 members were born, but only 78 are now resident therein; in the South Atlantic, South Central and Rocky Mountain States, 44 members were born, but only 19 now live there. In 17 states no Academy member resides; in these states, however, 73 members were born. In seven states no member was born; in five of these no member now resides. The trend of Academy members is naturally toward the centers of learning and of active research in science.

Seventy-eight Academy members were born in foreign countries; four of these are of American parentage. Twenty-two are from Canada, England, Scotland, Ireland and India; 10 from Sweden, Norway and Denmark; 23 from Germany, Austria, Holland and Switzerland; 16 from Russia, Poland and Hungary. The number of Academy members whose native tongue is not English is 54, or 12.4 per cent of the total membership.

The data on distribution of the Academy membership in universities and research organizations, including the Government, are listed in table 2. Of the total membership, 315 (72.4 per cent) are connected with universities and colleges; 25 (5.7 per cent) serve in Government positions, while 62 (14.3 per cent) are connected with endowed research groups; and 33 (7.6 per cent) with state and industrial research laboratories.

Age Distribution of Membership.—Table 3 presents the numbers of members in each age group and in the various Academy sections at the time of election and on July 1, 1948; also the average age and median age of the members at election and on July 1, 1948. In the preparation of this table the age of each member, in years and tenths of a year, at the time of his election and on July 1, 1948, was first ascertained. To facilitate the computations the average date (April 27) of the annual Academy meeting, at which elections are held, was used. Various methods may be used for computing the age of a person to tenths of a year of 365 or 366 days and the results obtained

vary slightly with the method employed; but they are not significant in this case and may be neglected. The method adopted in this paper for computing the ages of members to tenths of a year may best be illustrated by an example; if the birthday of a member falls between April 27 and June 1 his age at election is, say, 41.9 years; whereas if it occurs between March 21 and April 26 of the same year his age is 42.0 years. With the aid

TABLE 2

DISTRIBUTION OF MEMBERSHIP IN UNIVERSITIES AND RESEARCH ORGANIZATIONS (JULY 1, 1948)

UNIVERSITIES

Harvard	51	Wisconsin	12	Iowa	4	Smith	1
California	34	Princeton	10	Virginia	4	Haverford	1
Chicago	27	Cornell	9	Indiana	4	Brooklyn	1
Columbia	25	Pennsylvania	8	Rochester	3	Missouri	1
Yale	18	Wash. U., St. Louis	8	Brown	2	Swarthmore	1
Cal. Tech.	17	Illinois	7	North Carolina	2	Rice Inst.	1
Stanford	17	Texas	5	U. of St. Louis	2	Tennessee	1
M.I.T.	13	Michigan	4	Northwestern	1	New York	1
Johns Hopkins	13	Minnesota	4	Tufts	1	New Mexico	1
						Utah	1

RESEARCH ORGANIZATIONS

U. S. Government:		Endowed and Industrial Research Groups:			
Geological Survey	8	Carnegie Inst. Wash.	27	Ortho Research Found.	1
Smithsonian	3	Rockefeller Inst	19	Research Corp.	1
Bur. Standards	3	Inst. Advanced Study	7	Wistar Inst.	1
Nat. Inst. Health	3	Am. Tel. and Tel.	6	Bethlehem Steel Co.	1
Atomic Energy Comm.	2	General Electric	6	Carbide & Carbon Chem.	1
Adv. Com. Aeronautics	2	Am. Mus. Nat. Hist.	2	E. I. du Pont de Nemours	1
Bur. Entomology	1	N. Y. Botan. Garden	2	General Motors	1
Bur. Plant Indust.	1	Conn. Agr. Station	2	International Nickel	1
Research and		Gulf Res. & Dev.	2	Lederle Laboratories	1
Development Board	1	A.A.A.S.	1	Merck and Co.	1
Weather Bureau	1	R. B. Jackson		Monsanto Chemical Co.	1
		Mem. Lab.	1	Pan Am. Petroleum	1
		Loomis Lab.	1	Radio Corp. America	1
		Lowell Observatory	1	Univ. Oil Products	1
		N. J. Agr. Exp. Sta.	1	Westinghouse Electric	1
		N. Y. State Museum	1	J. G. White Corp.	1

of a short table prepared on this basis it is a simple task to ascertain the age of a member to tenths of a year. The fact that Pearl in his article of 1925 used April 15 as the date of election of each member was recognized by him to be unimportant in his study of vital statistics of Academy members.

Table 4 is supplemental to table 3 and shows the range of ages at election (column I) and as of July 1, 1948 (column II), in each section. This table is of special interest because it indicates that workers in certain fields of

science, such as mathematics, physics, chemistry and astronomy (the physical sciences) develop and establish records of accomplishment earlier than is the case with workers in the sciences which are in part descriptive in character and require field work (geology, botany, engineering) or long experience in the behavior of living matter (biological sciences). In the case

TABLE 3

AGES OF MEMBERS AT ELECTION AND ON JULY 1, 1948

YEARS	NO. AT ELECTION	NO. ON JULY 1, 1948
30–34	9	0
35–39	40	7
40–44	91	34
45–49	98	47
50–54	96	43
55–59	52	68
60–64	30	69
65–69	17	56
70–74	2	51
75–79	0	31
80–84	0	20
85–89	0	5
90–94	0	4

	AVERAGE AGES OF SECTION GROUPS		
NAME OF SECTION	NO. IN SECTION	AGE AT ELECTION	AGE ON JULY 1, 1948
1. Mathematics	26	43.1	57.9
2. Astronomy	26	47.8	67.9
3. Physics	60	44.8	57.0
4. Engineering	35	52.5	62.4
5. Chemistry	58	47.3	57.8
6. Geology	38	54.0	67.2
7. Botany	35	50.8	62.3
8. Zoology	38	51.3	62.7
9. Physiology	45	49.7	59.1
10. Pathology	37	53.3	63.2
11. Anthropology	9	52.6	63.6
12. Psychology	23	47.9	62.0

Average age July 1, 1948	61.5	Median age July 1, 1948	61.1
Average age at election	49.2	Median age at election	48.9
Average age at election (1935)	48.9		

TABLE 4

RANGES OF AGES IN EACH SECTION AT ELECTION (COL. I) AND ON JULY 1, 1948 (COL. II)

SECTION	NO. IN SECTION	AGE RANGE IN 10-YEAR PERIODS: 30–40 (I)	(II)	40–50 (I)	(II)	50–60 (I)	(II)	60–70 (I)	(II)	70–80 (I)	(II)	80–90 (I)	(II)	90–100 (I)	(II)
1. Mathematics	26	9	1	14	7	3	6	0	8	0	3	0	1	0	0
2. Astronomy	26	4	0	12	2	7	3	3	7	0	12	0	2	0	0
3. Physics	60	14	1	34	25	9	10	3	13	1	8	0	3	0	0
4. Engineering	35	2	1	9	3	19	11	4	12	1	6	0	2	0	0
5. Chemistry,	58	10	3	28	12	16	19	3	16	0	4	0	4	0	0
6. Geology,	38	1	0	10	4	16	8	11	11	0	9	0	4	0	2
7. Botany	35	0	0	20	6	9	10	6	8	0	9	0	1	0	1
8. Zool. & Anat.	38	1	0	19	4	13	16	5	8	0	7	0	3	0	0
9. Physiol. & Bioch.	45	3	1	21	8	17	13	4	17	0	6	0	0	0	0
10. Pathol. & Bact.	37	1	0	9	5	22	8	5	15	0	7	0	2	0	0
11. Anthropology	9	0	0	3	1	5	3	1	2	0	3	0	0	0	0
12. Psychology	23	3	0	9	5	10	4	1	7	0	6	0	1	0	0
Total	430	48	7	188	82	146	111	46	124	2	80	0	23	0	3
Per cent		11	2	44	19	34	26	11	29	1	18	0	5	0	1

of psychology, a relatively new science, many workers are applying the methods of the physical and biological sciences to the study of the behavior of the human being and some of its younger workers have been elected to Academy membership. The last line of the table presents in percentage

form the distribution of ages at election and on July 1, 1948, of the section members in the ten-year ranges.

In table 5 the numbers of members, as of July 1, 1948, who were elected in a given year, are listed. In the table the effect of raising in 1942 the limit of the number of members who may be elected at any annual meeting is

TABLE 5

NUMBER OF PRESENT MEMBERSHIP WHO WERE ELECTED IN A GIVEN YEAR

YEAR	NO. ELECTED	YEAR	NO. ELECTED	YEAR	NO. ELECTED	YEAR	NO. ELECTED
1948	30	1938	15	1928	8	1918	7
1947	29	1937	13	1927	10	1917	7
1946	28	1936	11	1926	0	1916	4
1945	29	1935	10	1925	7	1915	3
1944	24	1934	11	1924	7	1914	1
1943	25	1933	12	1923	7	1913	4
1942	15	1932	11	1922	7	1912	1
1941	15	1931	9	1921	6	1911	1
1940	13	1930	11	1920	8	1910	3
1939	14	1929	11	1919	5	1909	1
						1908	2
Total	222		114		65		34

TABLE 6

NUMBER OF ACADEMY MEMBERS, AS OF JULY 1, 1948, IN EACH SECTION AND AGES OF ELECTION WITHIN EACH 10-YEAR RANGE. IN EACH GROUP COLUMN I LISTS NUMBER OF MEMBERS ELECTED BEFORE 1943; COLUMN II, THOSE ELECTED THEREAFTER

SECTION	AGE RANGE 30–40 (I)	30–40 (II)	40–50 (I)	40–50 (II)	50–60 (I)	50–60 (II)	60–70 (I)	60–70 (II)	70–80 (I)	70–80 (II)
1. Mathematics	6	3	10	4	3	0	0	0	0	0
2. Astronomy	4	0	11	1	7	0	2	1	0	0
3. Physics	11	3	16	18	5	4	1	2	0	0
4. Engineering	1	1	5	4	10	9	3	1	1	0
5. Chemistry	5	5	18	10	5	11	0	3	1	0
6. Geology	1	0	6	4	9	7	7	4	0	0
7. Botany	0	0	10	10	7	2	3	3	0	0
8. Zool. & Anat.	1	0	12	7	6	7	4	1	0	0
9. Physiol. & Bioch.	1	2	13	8	6	11	0	4	0	0
10. Pathol. & Bact.	1	0	6	3	13	9	0	5	0	0
11. Anthropology	0	0	3	0	3	2	0	1	0	0
12. Psychology	2	1	6	3	8	2	0	1	0	0
Total	33	15	116	72	82	64	20	26	2	0
Per cent	13	8	46	41	32	36	8	15	1	0

clearly shown. In this list the members emeriti are included. The list shows that the period of service of 222 members is from 0 to 10 years; of 114 members, 10 to 20 years; of 65 members, 20 to 30 years; of 32 members 30 to 40 years; and of 2 members, in excess of 40 years.

In table 6 the numbers (as of July 1, 1948) of members elected prior to

1943 are listed in the first (I) of two columns in each section and for each ten-year range of age at election; in the second column (II) similar data for numbers of section members elected to membership between 1943 and 1948 are presented. At the foot of each column its sum is given; the total number of section members elected before 1943 is 253; that for members elected thereafter is 177. In the last line the ratios are expressed in per-

TABLE 7

AGES AT ELECTION FOR THE PERIODS: (I) 1863–1924; (II) 1925–1942; (III) 1943–1948; (IV) 1863–1948

ELECTED: AGE	1863–1924 (I)	1925–1942 (II)	1943–1948 (III)	1863–1948 (IV)	AGE	1863–1924 (I)	1925–1942 (II)	1943–1948 (III)	1863–1948 (IV)
26–27	1	0	0	1	56–57	17	8	9	34
28–29	1	0	0	1	58–59	17	10	9	36
30–31	4	1	0	5	60–61	7	9	8	24
32–33	7	4	0	11	62–63	16	12	4	32
34–35	15	4	2	21	64–65	6	11	4	21
36–37	29	5	5	39	66–67	6	5	4	15
38–39	25	11	6	42	68–69	7	2	5	14
40–41	29	17	13	59	70–71	3	3	0	6
42–43	28	9	17	54	72–73	3	1	0	4
44–45	39	17	16	72	74–75	2	0	0	2
46–47	32	18	11	61	76–77	0	0	0	0
48–49	35	22	13	70	78–79	0	0	0	0
50–51	38	38	15	91	80–81	0	1	0	1
52–53	27	14	16	57	82–83	1	0	0	1
54–55	26	16	12	54					
					Total	421	238	169	828

centages; thus in column I of the age range 30–40 the number 33 is 13 per cent of 253; the number 15 of column II is 8 per cent of 177. Comparison of the percentages shows perceptible shifts in the age groups, 30–40 and 60–70, as a result of the change from 15 to 30 in the limit of the candidates who may be elected each year. The elections of 1943 to 1948 are notable for their low percentage in the lower and upper age groups. Too great conservatism with respect to the lower age group may be just as bad as too little and may reduce the effectiveness of the Academy, especially in its relation to Government problems. Younger men have the energy, enthusiasm and capacity to undertake the solution of a given problem and to do this in addition to their ordinary tasks. Many men on reaching 70 are less active than in earlier years and cannot undertake large tasks. In the Academy the number of members past 70 is 106, in other words nearly one quarter of the entire membership has attained the usual retiring age. The number of Academy members whose ages exceed 60 is 230, a number greater than one-half the Academy membership.

Of the 407 members elected during the period 1925–1948 the number

living on July 1, 1948, was 359; of these, the entire number (38) of members whose age at election was less than 40 is still living. The average age at election of the 359 members is 50.0. The average election age of the 47 members who have died is 58.7 years; the list includes 3 members elected at the ages: 80.2, 72.5 and 70.1, respectively; 20 between 60 and 70; 17 between 50 and 60; and 6 between 40 and 50.

Vital Statistics of Members Elected during the Period 1925–1948.—Comparison can now be made of the ages of members elected during the years 1925 to 1948 with those of the 421 members elected during the period 1863–1924, including charter members, as reported upon by Pearl in 1925. The necessary data are presented in tables 7 and 8. In table 7 the numbers of members elected in each age group are listed for the four periods: 1863–1924 (column I as given by Pearl); 1925–1942 (column II); 1943–1948 (column III); and 1863–1948 (column IV). In each column all members (living, resigned and deceased) who were elected during the given period are included. If a graph be prepared, one curve for each column I, II, III, in-

TABLE 8

VITAL STATISTICS ON AGES AT ELECTION

PERIOD	MEAN AGE	MEDIAN AGE	PER CENT UNDER 40	NO. ELECTED	AVERAGE EACH YEAR
Charter 1863	51.7	51.3	16.7	48	..
1863–1883	44.5	41.3	45.5	95	5
1884–1904	46.5	45.1	20.0	65	3
1905–1924	50.5	49.5	8.4	213	11
1925–1942	51.0	50.6	10.5	238	13
1943–1948	50.6	50.2	7.7	169	28

spection of the curves shows that the tendency to elect older men has increased in the course of the years. The increase may be connected with that of the life span during the past several generations; to the membership, which now elects the new members, a man of 40 may appear to be of the same degree of youngness as did a man of 30 or 35 two or three generations ago. Another factor in favor of the rise may be the present tendency toward regimentation and coöperative effort in solution of problems under group attack. At present this is felt in all walks of life and may tend to retard recognition of contributions by young investigators and to have a lag effect on the ages at election of the Academy membership.

In 1945 Edwin B. Wilson compared the results obtained by Pearl on the ages at election for certain periods covering the years 1863 to 1924 with those for the years 1943 to 1945. He concluded with Pearl "that the tendencies toward a higher mean age, a median nearer the mean, . . . have persisted and been intensified." On the basis of the data considered by Wilson and presented in his table, this conclusion was correct and justified. However, as shown in table 8, the addition of data for the periods 1925–1942

and 1943–1948 indicates that for the periods 1905–1924, 1925–1942, 1943–1948, the average ages at election, the percentages of younger men elected at ages under 40 and of older men elected at 62 and at a later age have remained nearly constant. Thus, for the entire period 1905–1948, the average age at election was 50.8; for the years 1904–1924 it was 50.5; for 1925–1942, 51.0; and for 1943–1948, 50.6. The ratios of men under 40 elected during the same periods were, respectively: 56 to 620 or 9.0 per cent; 18 to 213 or 8.5 per cent; 25 to 238 or 10.5 per cent; and 11 to 169 or 7.7 per cent. Similarly for men 62 and older at election the numbers for the same periods were, respectively: 75 to 620 or 12.1 per cent; 23 to 213 or 10.8 per cent; 35 to 238 or 14.7 per cent; 17 to 169 or 10.1 per cent.

These results indicate that for the past 44 years the Academy membership with respect to ages at election has changed but little. For shorter periods, such as a single year or only a few years the ratios may vary appreciably. Thus for the 6-year period 1943–1948 a slight decrease in the average age and a perceptible decrease in the percentage of the men elected under 40 are noticeable. In spite of the small fluctuations the figures indicate that the average age at election and the percentages of men elected at ages under 40 and 62 or older have remained reasonably constant; in other words, the Academy membership has continued to be stable and in balance.

THE MIOCENE OCCURRENCE OF SEQUOIA AND RELATED CONIFERS IN THE JOHN DAY BASIN

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Read before the Academy, November 16, 1948

Four genera of the family Taxodiaceae—Sequoia, Taxodium, Metasequoia and Glyptostrobus—are among the most numerous and widely distributed conifers in the Tertiary record of Western North America. All of them have been found in rocks of Middle Miocene age in the John Day Basin and adjacent areas of eastern Oregon, although none of them live there today. Taxodium and Metasequoia occur together in the Mascall flora, and the other two are members of related floras located within sixty miles.[1] By contrast the distribution of existing trees of these genera shows no overlapping ranges; Sequoia and Taxodium are confined to North America while Metasequoia and Glyptostrobus now live only in Asia. *Sequoia sempervirens* is found along the Pacific Coast from central California to Oregon;[2] *Taxodium distichum* occupies the Atlantic coastal plain from Delaware south to Florida, and around the Gulf to the Mexican border,

with a northward extension up to the Mississippi valley to southern Illinois and Indiana; a second species, *T. mucronatum*, extends from northern Mexico into Guatemala. *Metasequoia glyptostroboides* is known to occur in a limited area in central China, near the Szechuan-Hupeh border; *Glyptostrobus pensilis* has a restricted distribution in the lowlands of southern China.[3] It is apparent that during the approximately twenty million years since the Middle Miocene, events have transpired, changes have occurred, which have eliminated all of these conifers from the John Day Basin, and have greatly restricted their distribution elsewhere.

All four of the genera under consideration have been recorded from the Cretaceous[4] and Eocene[5] deposits of Alaska, or from Greenland, Iceland, Spitzbergen, arctic Siberia and the islands north of Canada.[6] At these northern localities they are associated with hardwood-deciduous genera such as Acer, Alnus, Betula, Carpinus, Castanea, Cercidiphyllum, Fagus, Liquidambar, Quercus and Ulmus. This mixed conifer-hardwood assemblage, which appears to have had its origin at high northern latitudes, is known as the Arcto-Tertiary Flora. Up to the close of the Eocene epoch, approximately forty million years ago, mild climates made possible the existence of these trees far beyond their present northern limits. At about that time there is evidence of a progressive change toward lower winter temperatures which continued down to the Pleistocene, resulting in restriction of the ranges of most of these trees at the north, and their extension into lower latitudes. By the close of the Oligocene epoch, some ten million years later, the Arcto-Tertiary Flora was widely established at middle latitudes, where the fossil record of its presence is known from Oregon, Manchuria and elsewhere on both sides of the Pacific. *M. heerii* was the dominant conifer of this forest in eastern Oregon, and judging from the habits of its living equivalent, *M. glyptostroboides*, in central China, it occupied moist valleys and ravines at moderate elevations, under climatic conditions characterized by humid summers and mild winters.[7] *T. dubium* was an uncommon associate, as judged by its fossil representation; this tree, like its living equivalent, may be considered to have lived in valley swamp habitats. The other two related genera are not known to have lived nearby, though *S. affinis* is of common occurrence in Oligocene deposits near the coast of Oregon and Washington, and both it and *G. oregonensis* have been recored from localities farther inland.

By the middle of the Miocene epoch which followed, changes were taking place which had a profound effect upon the topography and climate of the John Day Basin and upon its vegetation. The Cascade Range was being built up by lava flows and accumulations of pyroclastic materials,[8] and streams flowing westward across its axis were being dammed or diverted. Local eruptions in eastern Oregon were further altering the drainage, producing many small lakes and swamps along the stream courses.

Deposits of volcanic ash, which make up so conspicuous a part of the Mascall formation, were accumulating in these basins, and were enclosing one of the most complete known records of the vegetation of the past. Along the poorly drained valleys of the John Day area, the swamp cypress, *T. dubium*, took the place of *M. heerii* as the dominant conifer, together with a group of hardwoods whose living equivalents are now associated with the swamp cypress in the eastern United States, including such genera as Acer, Betula, Carya, Nyssa, Platanus and Quercus. Metasequoia persisted as a member of the forest on higher ground around the borders, as judged by the occurrence of its leafy shoots and cones in relatively small numbers in the fossil record.

From two localities in the Blue Mountains, some 40 miles to the east, a flora has been collected which shows many species in common with the Mascall flora, but which includes a group of upland trees not therein represented.[9] From this Blue Mountains flora both *T. dubium* and *M. heerii* are missing. The common occurrence of Sequoia, a genus now confined to Coastal valleys and slopes up to elevations of 3000 feet, in the interior of Oregon is not fully explained; it has been recorded also from somewhat older floras still farther inland, the Missoula flora of Montana, the Intermediate flora of Yellowstone Park, the Florissant flora of Colorado, among groups of plants which appear to have occupied upland environments. Apparently Sequoia was finding it difficult to survive under the new climatic regime, and prior to its extinction in the interior was making a last stand at higher elevations where there was adequate precipitation. The ranges of several existing species of conifers suggest a similar contemporary adjustment to dry interior conditions; *Abies grandis*, *Pinus contorta*, *Pseudotsuga taxifolia*, *Taxus brevifolia*, *Thuja plicata* and *Tsuga heterophylla* show discontinuous distribution from the Pacific Coast into the mountains as far east as Montana; *Chamaecyparis lawsoniana*, *C. nootkatensis* and *Picea sitchensis* extend eastward for lesser distances. The environment of the Blue Mountains flora, as shown by Sequoia and other conifers, was clearly of a more upland character than that of the Mascall flora, with its dominant Taxodium and associated swamp types of hardwoods.

The occurrence of *G. oregonensis* as the dominant conifer of the Stinking Water flora, in the Stinking Water Basin, 34 miles east of Burns on the south side of the Blue Mountains, is somewhat difficult to interpret, especially since there is little available information regarding the natural environment of trees of this genus in southern China. The water pine, *G. pensilis*, is reported to grow on open stream courses and sandy delta margins near sea level. It appears improbable that a similar coastal environment was available in the Stinking Water Basin during the Miocene epoch, at which time the nearest recorded strand was nearly 300 miles to the west. The general aspect of the Stinking Water flora, and of the Beulah and Suc-

cor Creek floras immediately to the east, both of which also include Glyptostrobus, suggests more open country than that in the John Day Basin, with probably much less rainfall. On this terrain, largely occupied by an oak woodland, the Miocene water pines appear to have lived along the borders of large streams, in whose deposits their leafy shoots and cones have accumulated and have been preserved in great abundance.

Although the proximity of the four taxodiaceous genera during the Miocene is not wholly explained by these suggestions, it seems apparent that each of them occupied a distinctive ecologic niche, and that the topography in this part of eastern Oregon was of a varied character, with swampy valleys, cool moist slopes and broad, open savannas traversed by large streams. The deciduous habit of all except Sequoia, together with the predominantly deciduous habit of the hardwood associates, may be interpreted as indicating a summer-wet climate, and a cold though probably not severe winter. In preceding Eocene time, before the Cascade Range had reached sufficient height to constitute a climatic barrier, the subtropical type of vegetation which characterized the coastal margin as known from the Goshen and Comstock floras was also represented in the Clarno flora of the John Day Basin. During the Oligocene epoch which followed, there was already apparent a differentiation between the coastal vegetation, with Sequoia, and that of the interior, with Metasequoia. The Miocene epoch shows still further changes in the interior, with a trend toward the continental type of climate which now characterizes this region. The rich mixed forest of conifers and hardwoods gradually disappeared as the Cascade barrier to the west continued to rise, reducing precipitation to its present level of some eleven inches a year in the John Day Basin, and bringing lower winter temperatures. By Pliocene time only the hardier members of the Arcto-Tertiary Flora were living east of the Cascades; today a few of them—birch, cherry, fir, maple, pine and poplar—remain at higher levels or along streams where there is a permanent supply of water. Sequoia, Taxodium, Metasequoia and Glyptostrobus have long since disappeared.

With the exception of Sequoia, all of these genera now live in regions with summer rainfall; heavy summer fogs supplement the moisture requirements of Sequoia, which lives along the Pacific Coast in a region of winter rainfall. Mild temperatures characterize the habitats of all four genera; although Taxodium ranges north up the Mississippi Basin to southern Illinois, where winter temperatures fall well below freezing for several months, the swamp cypress attains its largest size and greatest abundance in the southerly portion of its range. It seems probable that throughout its history the Arcto-Tertiary Flora, made up largely of deciduous conifers and hardwoods, has been largely confined to areas characterized by summer rainfall, and that it has had its best development where seasonal changes in temperature have not been extreme. Elements of this Flora are still widely

distributed at middle latitudes in the northern hemisphere, but only locally, where their delicately balanced climatic and topographic requirements are fully met, have Sequoia, Taxodium, Metasequoia and Glyptostrobus survived. The continued existence of these genera is dependent upon the extent of further environmental changes, as well as upon man's wisdom in conserving them.

[1] A study of the Miocene floras of the Columbia Plateau, by D. I. Axelrod and the writer is nearing completion.

[2] A related genus, Sequoiadendron, occurs in the Sierra Nevada of California.

[3] Other genera of the Taxodiaceae. Mention of other genera of the Taxodiaceae is here omitted, since none of them is known to occur in abundance in the fossil record.

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[5] Hollick, A., *U. S. Geol. Surv.*, Prof. Paper 182, 1936.

[6] Heer, O., *Flora Foss. Arct.*, 7 vols. (1868–1883).

[7] Chaney, R. W., these PROCEEDINGS, **34**, 503–515(1948).

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A NOTE ON CENTROMERE ORGANIZATION

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The centromere or kinetochore is that portion of the chromosome concerned with congression to and orientation on the spindle at metaphase as well as with the more dramatic poleward movement at anaphase. In the great majority of plants and animals the centromere is a sharply delimited region of the chromosome differing in its morphology, chemical constitution and function. During anaphase the centric region leads the poleward advance since the half-spindle components which bring about the movement are elaborated solely by the localized centromere, and the chromosome arms play no active rôle, being passively pulled along the spindle to the poles. Acentric fragments form no spindle fibres and their behavior on the spindle figure is erratic.

There exist, however, chromosomes with half-spindle fibres originating not solely from a localized region of the chromosome but from the whole length of the chromosome. In these chromosomes the entire poleward face of the chromosome is involved in fibre formation and a solid sheet of spindle fibres is produced. Schrader[1] found that this diffuse type of centromere characterized the chromosomes of the hemipterous and homopterous in-

sects. Chromosomes with diffuse centromeres are oriented on the spindle so that the body of the chromosome lies in a plane perpendicular to the longitudinal axis of the spindle, and the separating chromosomes maintain a position parallel to the spindle plate, although in some species the ends of the chromosomes are bent towards the pole (this is clearly an adaptation allowing the body of the chromosome to converge to the pole of the spindle). As Hughes-Schrader and Ris[2] and Ris[3] have shown, the behavior of fragments derived from chromosomes with diffuse centromeres is different from that of those derived from chromosomes with localized centromeres. They observed that these fragments were oriented on the metaphase spindle in a normal manner; each fragment formed spindle fibres along its length and had the broad-side orientation on the spindle characteristic of intact chromosomes. These fragments behaved normally in a number of mitoses. It appears in this type of centromere that the ability to form spindle fibres is shared by all regions of the chromosome—i.e., the centromere is spread or diffused continuously along the chromosome.

In addition to chromosomes with single, localized centromeres and those with diffuse centromeres is the polycentric type found in the germ line of *Ascaris megalocephala* where the multiple or compound chromosomes consist of an end-to-end union of smaller chromosomes each possessing a localized centromere. The anaphase movement of the compound chromosomes is the resultant of the action of the individual centromeres. White[4] found that experimentally induced fragments of these compound chromosomes were normally oriented on the spindle. The polycentric and diffuse types have a superficial similarity but they differ in a fundamental manner since the centromere substance is believed to be spread continuously along the chromosome in diffuse centromeres while the polycentric chromosomes possess many small individual centromeres. Presumably every fragment, no matter how minute, from a chromosome with a diffuse centromere would possess centric material and be able to form fibres while polycentric chromosomes could not be subdivided indefinitely into normally behaving fragments.

Among plants chromosomes with other than localized centromeres have been rarely observed, but Malheiros, de Castro and Câmara[5] found non-localized centromeres in the rush *Luzula purpurea*. It is not clear whether these chromosomes are polycentric or have diffuse centromeres, but a polycentric condition appears much more likely. Further, there are, both in maize and rye, strains in which regions of the chromosomes other than the true centromere form spindle fibres (Rhoades and Vilkomerson,[6] Prakken and Müntzing,[7] and Östergren and Prakken[8]). This unusual behavior is restricted to the two meiotic divisions, and in maize we have been able to demonstrate that the formation of these subsidiary spindle fibres is a product of the activity of the true centric region (Rhoades[9]).

There is also the dicentric condition reported by Piza[10] for the chromosomes of the Brazilian scorpion *Tityus bahiensis* (2n = 6) where, according to Piza, each of these large chromosomes has a terminal centromere at both ends. He came to this conclusion because the two ends of the chromosomes lead the way to the pole even though spindle fibres are present along the entire poleward face of the chromosome. These interjacent fibres he believed arose from some influence emanating from the two end centromeres and moving along the body of the chromosome. Piza also holds that the diffuse centromeres of the insect orders studied by Schrader are of the same dicentric type as in Tityus since in some of these forms the ends also pass to the poles in advance of the rest of the chromosome. However the end centromeres postulated by Piza have never been observed and there are reasons for believing that the Tityus chromosomes are polycentric. Whether or not these chromosomes of the scorpion are dicentric or polycentric can be determined by a simple experimental procedure—i.e., the fracturing of the chromosome by short-wave irradiation. We, therefore, treated nine Tityus males with x-rays, their testes were removed at various times following treatment, fixed in Dusbock-Brasil fluid, sectioned and stained with hemotoxylin.* The discrimination between the dicentric and polycentric condition can be obtained from the behavior of fragments with recently broken ends. As Piza has correctly pointed out, the ends of the chromosom espass to the pole precociously (this is first evident at late metaphase); this he attributes to the terminal centromeres. However, the turning of the ends towards the poles may be nothing more than an adaptation permitting the chromosomes (huge compared to cell size) to converge to the spindle poles and they may be polycentric. If the intact chromosomes possess two terminal centromeres then fragments would at most have only one normal end while the other end must be a freshly broken one; it would be expected then that the normal end would continue to show a precocious poleward movement while the freshly broken end would have an anaphase migration no different from the rest of the chromosome. On the

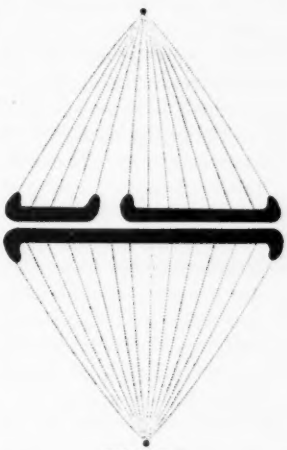

FIGURE 1

A bivalent at late metaphase in a primary spermatocyte of *Tityus bahiensis*. The upper homolog has been fractured as the consequence of x-ray irradiation. The two freshly broken ends show the same precocious poleward movement as do the normal ends.

other hand, if these scorpion chromosomes are polycentric the old and new ends might both show the same early poleward advance and no difference in behavior on the spindle would be manifest. This is precisely what was observed in several score of primary and secondary spermatocytes with fragmented chromosomes. One such case is illustrated in figure 1. There is, in our opinion, no convincing evidence that Tityus has dicentric chromosomes. Rather the behavior of the chromosome fragments reported above together with the fact that close relatives of Tityus have numerous small chromosomes with localized centromeres and that the postulated terminal centromeres have never been observed make it appear highly probable that these chromosomes are polycentric.

Centromeres have been classed as either localized or non-localized. The diffuse centromere found in the hemiptera and homoptera, where it is apparently spread continuously along the chromosome, is the best example of the non-localized type. The proper classification of polycentric chromosomes is somewhat uncertain on this basis. Their appearance and behavior on the spindle is hardly distinguishable from that of the hemipteran chromosomes with diffuse centromeres yet these polycentric chromosomes are constituted by an end-to-end union of numerous small chromosomes each with its own individualized centromere. Chromosomes with localized centromeres such as are found in the great majority of organisms are monocentric. The evidence for dicentric chromosomes in Tityus is unconvincing; the experimental evidence suggests they are polycentric. It is to be questioned if dicentric chromosomes ever are found in any normal chromosome complement. Of some interest in connection with the problem of the evolution of centromere form and function is the situation obtaining in those strains of maize and rye where normally monocentric chromosomes have subsidiary fibres at regions devoid of centromeres. However, in maize it has been ascertained that the formation of these extra fibres is controlled, in part at least, by the localized centromere.

* We wish to express our gratitude to Professor André Dreyfus of the University of São Paulo, São Paulo, for making the arrangements necessary for the x-ray treatments and to Professor F. G. Brieger for providing facilities.

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PRIMARY IDEALS IN GROUP ALGEBRAS*

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Let G be a locally compact abelian group and A its L_1-algebra, that is, the algebra of all complex functions summable with respect to Haar measure, with convolution as multiplication. Let I be a closed ideal in A. A question stemming from Wiener's Tauberian investigations is the following: is I the intersection of the regular maximal ideals containing it? L. Schwartz[1] has recently published a counter-example. This negative result increases the interest in special cases where the answer is affirmative. Ditkin,[2] Beurling[3] and Segal[4] have obtained an affirmative answer in the following case: I is a primary ideal (that is, it is contained in exactly one regular maximal ideal), and G is the group of real numbers or the group of integers. The purpose of this note is to extend this result to any locally compact abelian group. This will be accomplished by the methods of Segal, together with the structure theorems for groups. The essential idea is that the discrete case can be treated "elementwise," and thus reduced to the case of a cyclic group.

THEOREM 1. *In the L_1-algebra of a locally compact abelian group, any closed primary ideal is maximal.*

In the proof we shall use the following lemma, which is a slight extension of Lemma 2.7.1 in S.

LEMMA 1. *Let R be the direct sum of a finite number of groups, each of which is the group of real numbers or of integers. Then in $L_1(R)$ there exists a bounded sequence f_n such that (1) the Fourier transform of each f_n is 1 in a neighborhood of 0 (the neighborhood depending on n), (2) $f_n g - f_n \int_R g \to 0$ as $n \to \infty$, for any g in $L_1(R)$.*

Any locally compact abelian group G can be written as the direct sum of a vector group V and a group H having a compact open subgroup C. We use the additive notation, and the symbol O will denote the identity element of any group. Also we shall write (f, g) for the element of $L_1(G)$ which is the product of f in $L_1(V)$ and g in $L_1(H)$. Let I be a closed primary ideal in $L_1(G)$ and M the unique regular maximal ideal containing it. Our task is to prove $I = M$. We may suppose without loss of generality that the character associated with M is O. The following lemma constitutes the essential step in the proof.

LEMMA 2. *Let e be the characteristic function of C, and let p, q be arbitrary elements in $L_1(V)$, $L_1(H)$; write $c = \int_H q$. Then $(p, q - ce)$ is in I.*

To prove this, we begin by observing that we can write q as a sum of functions each confined to a single coset of C; consequently it will suffice

to consider the case where q itself vanishes outside the coset $h + C$. We now pause to apply Lemma 1 to the case of the L_1-algebra of an infinite cyclic group, generated say by h'. For any ϵ, this provides us with an element z' satisfying

$$\| h'z' - z' \| < \epsilon, \tag{1}$$

and such that the Fourier transform of z' is 1 in a neighborhood of O. Suppose z' is equal to a_n at the coördinate nh'. We return to $L_1(H)$ and there define z to be equal to a_n on the coset $nh + C (n = 0, \pm 1, \ldots)$ and otherwise 0.[5] Then the Fourier transform of z is again 1 in a neighborhood of 0. Moreover if we denote the characteristic function of $h + C$ by f, the inequality (1) may be transferred to read $\| fz - z \| < \epsilon$ (we are assuming the measure of C to be 1). It can be verified that $qz = cfz$. Now $(p, q - qz + cz - ce)$ is in I, since[6] its Fourier transform vanishes in a neighborhood of 0. As $\epsilon \to 0$, $qz - cz \to 0$. Since I is closed, we deduce that $(p, q - ce)$ is in I.

We now complete the proof of Theorem 1. Let m denote any element in $L_1(G)$ whose Fourier transform vanishes at 0, that is $\int_G m = 0$. We have to prove that m is in I. We may choose $s_i, t_i (i = 1, \ldots, r)$ in $L_1(V)$, $L_1(H)$ such that

$$\| m - \sum (s_i, t_i) \| < \epsilon. \tag{2}$$

Write $\int_V s_i = a_i$, $\int_H t_i = b_i$; then

$$\left| \sum a_i b_i \right| < \epsilon. \tag{3}$$

Choose $f_n \epsilon L_1(V)$ as in Lemma 1, with n so large that

$$\| s_i f_n - a_i f_n \| < \epsilon / \sum \| t_i \| \tag{4}$$

for each i. By Lemma 2,

$$(a_i f_n, t_i - b_i e) \epsilon I. \tag{5}$$

Finally,

$$(s_i f_n - s_i, t_i) \epsilon I, \tag{6}$$

since its Fourier transform vanishes in a neighborhood of 0. Combining (2)–(6), we find that m differs from an element of I by less than $\epsilon(3 + \| f_n \|)$. This concludes the proof of Theorem 1.

Remark: If Lemma 1 were valid in H as well as in V, it would be quite simple to extend it to G. However, I have been able to do this only on the assumption that H/C is finitely generated. A straightforward argument then leads to the following result.[7]

THEOREM 2. *Suppose, in the above notation, that H/C is finitely generated.*

Let I be a closed ideal in $L_1(G)$ such that the set $\{M_t\}$ of regular maximal ideals containing I has a countable frontier. Then $I = \cap M_t$.

Applications: (1) Godement[8] states that, on the basis of Theorem 1, it is possible to extend Beurling's[3] results to any locally compact abelian group.

(2) Let T be an operator in a Banach space such that $\| T^n \| < K$ for all (positive and negative) n. We may show that T resembles a normal operator in Hilbert space to the following extent: if T has only 1 in its spectrum, then T is the identity operator.[9] To prove this, one considers the homomorphism of the L_1-algebra of the integers given by $\{a_n\} \to \sum a_n T^n$. The kernel is easily seen to be a closed primary ideal. Its maximality gives the desired result.

(3) The preceding example in effect discussed a representation of the group of integers. More generally, Godement[8] studies representations of any locally compact abelian group by operators in a Banach space; assuming Theorem 1, he proves that the only irreducible ones are one-dimensional. The following more general result is also a corollary of Theorem 1.

THEOREM 3. *Let G be the direct sum of a compact group C and a locally compact abelian group D. Then any irreducible representation of G by operators in a Banach space is finite-dimensional.*

To prove this, we consider the induced representation of D and its extension (S, p. 79) to $L_1(D)$. The kernel of this extension can be seen to be a primary ideal. By Theorem 1, the latter is maximal. Hence the elements of D are represented by scalar operators. This shows that the induced representation of C is also irreducible, from which it follows that the representation is finite-dimensional.[10]

As a corollary of Theorem 3, we note that if P is a primitive ideal in $L_1(G)$, then $L_1(G)/P$ is finite-dimensional and hence regular maximal; this proves Theorem 1.7 of S in a somewhat sharper form. We thus have a natural one-one correspondence between irreducible representations of G and primitive ideals in $L_1(G)$. With this result in hand, we may proceed down the list of the standard theorems known to hold in the abelian and compact cases. In particular, Theorem 1 of this paper extends to the direct sum of an abelian and a compact group.

* The author is a fellow of the John Simon Guggenheim Memorial Foundation, on leave from the University of Chicago.

[1] *C. R. Acad. Sci. Paris*, **227**, 424–426 (1948).

[2] *Uchenye Zapiski Moskov, Gos. Univ. Mat.*, **30**, 83–130 (1939).

[3] *Acta Math.*, **77**, 127–136 (1945).

[4] *Trans. Am. Math. Soc.*, **61**, 69–105 (1947). This paper will be cited as S.

[5] We are considering explicitly only the case where the coset $h + C$ is of infinite order in H/C; the case where it is of finite order is considerably simpler.

[6] S, Theorem 2.2.

[7] For the groups of Lemma 1, this extension of Theorem 1 is implicit in the work of Ditkin and Segal, and is mentioned explicitly by Schwartz.

[8] *Ann. École. Norm.*, **74**, 119–138 (1946). Theorem 1 is hypothesis A on page 136.

[9] For a reflective Banach space, this theorem can also be deduced from results of Lorch, *Trans. Am. Math. Soc.*, **49**, 18–40 (1941), and a stronger result can also be proved by using a theorem of Gelfand and Hille.

[10] It is well known that any irreducible representation of a compact group by unitary operators in Hilbert space is finite-dimensional. The methods which prove this can be extended to the case of a representation on any Banach space.

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*INTEGRAL REPRESENTATIONS OF BILINEAR FUNCTIONALS**

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1. Introduction.—Let L_2 designate the linear space of measurable functionals x with values $x(s)$ defined over the interval [0, 1] and with a pseudo-norm

$$|x|_{L_2} = [\int_0^1 x^2(s)\, ds]^{1/2}.$$

We term $|x|_{L_2}$ a *pseudo-norm* rather than a norm, since we do not find it convenient to require that x be the null element in L_2 when $|x|_{L_2} = 0$. The null element in L_2 shall instead be the functional with null values. The space C is that defined by Banach (p. 11). Let A be C or L_2 and B be C or L_2. A functional F is *bilinear* over $A \times B$ by definition if $F(x, y)$ is defined for each x in A and y in B, if the functionals $F(x, \cdot)$ and $F(\cdot, y)$, defined by F for fixed x and y, respectively, are additive and homogeneous over B and A, and if $|F(x, y)| \leqq |x|_A |y|_B M$ for some constant M and every (x, y) in $A \times B$.

The Space A: With $A = C$ or L_2 we shall define a space A_d (space of distribution functions) associated with A. Let g map an axis of reals R_1 into R_1. Let g be termed *canonical* rel. to [0, 1] if $g(s) = 0$ for $s \leqq 0$ and $g(s) = g(1)$ for $s \geqq 1$, and if g is left continuous except at most at $s = 1$. The space A_d shall be the linear space of functionals g canonical rel. to [0, 1], with a finite total variation $T(g)$ and norm (in the sense of Banach, p. 53) $|g|_{A_d} = T(g)$ when $A = C$; and with a finite Hellinger integral [cf. Hobson, Vol. 1, p. 673] when $A = L_2$ and a norm

$$|g|_{A_d} = \left[\int_0^1 \frac{[dg]^2}{ds}\right]^{1/2} \tag{1.1}$$

The Variation $h(A, B, k)$: We admit partitions π of [0, 1] into intervals $[0, s_1)[s_1, s_2) \ldots [s_{r(\pi)-1}, 1]$ open at the right when the parenthesis) is used. An s-function (step-function) η *associated* with π and *belonging* to

$[0, 1]$ by definition maps R_1 into R_1 with $\eta(s) = 0$ for s not in $[0, 1]$ and $\eta(s) = \eta_r$, a constant on the rth interval of π. Let (s, t) be Cartesian coördinates in a plane R_2. Let E' and E'' respectively represent the interval $[0, 1]$ on the s- and t-axes. Let π' represent the above partition of E' and let π'', with vertices $0 = t_0 < t_1 < \ldots < t_{n(\pi'')} = 1$ be a similar partition of E''. Let η and ζ be s-functions associated respectively with π' and π''. Set

$$\Delta_{rn}(k) = k(s_r, t_n) - k(s_{r-1}, t_n) - k(s_r, t_{n-1}) + k(s_{r-1}, t_{n-1}) \qquad \begin{cases} r = 1, \ldots r(\pi') \\ n = 1, \ldots n(\pi'') \end{cases} \tag{1.2}$$

for any k mapping R_2 into R_1. Let $A \times B$ be any one of the products $C \times C$, $C \times L_2$, $L_2 \times L_2$. We say that k is canonical rel. $E' \times E''$ if $k(s, \cdot)$ is canonical rel. E'' for each fixed s, and $k(\cdot, t)$ is canonical rel. to E' for each fixed t. With k canonical rel. $E' \times E''$ set

$$h(A, B, k) = \sup_{\eta, \zeta} \eta_r \zeta_n \Delta_{rn}(k) \tag{1.3}$$

summing as to r and n as in tensor algebra, and taking the sup. over all η and ζ for which $|\eta|_A \leqq 1$, $|\zeta|_B \leqq 1$. We understand thereby that $|\eta|_C = \text{max. } |\eta(s)|$. If $h(A, B, k) < \infty$, k is termed a d-function rel. $A \times B$. A principal theorem follows. See MT (2), Theorem 12.1 and Lemma 12.3, MT (3), Theorems 3.3 and 4.3.

THEOREM 1.1. *A functional ϕ bilinear over $A \times B$ has a value for $(x, y) \in A \times B$ given by a repeated L-S-integral*

$$F(x, y, k) = \int_0^1 x(s)\, d_s \int_0^1 y(t) d_t k(s, t) = \int^1 y(s) d_s \int_0^1 x(t) d_t k(t, s), \tag{1.4}$$

where k is a d-function rel. $A \times B$ uniquely determined by ϕ and

$$h(A, B, k) = \sup_{x, y} \frac{|\phi(x, y)|}{|x|_A |y|_B} \qquad \{|x|_A > 0,\ |y|_B > 0\}. \tag{1.5}$$

Conversely if k is a d-function rel. $A \times B$, the integrals (1.4) exist and define a functional ϕ bilinear over $A \times B$, for which (1.5) holds.

We shall need the following, see MT (2), Lemma 5.1 and Theorem 5.2, MT (3), Theorems 3.3 and 4.3.

THEOREM 1.2. *A functional g mapping R_1 into R_1 and canonical rel. $[0, 1]$ is in A_d if and only if sup. $|\eta_r \Delta_r(g)| < \infty$ taking the sup. as in (1.3) [with $\Delta_r(g) = g(s_r) - g(s_{r-1})$]. If g is in A_d*

$$\sup_{\eta} |\eta_r \Delta_r(g)| = |g|_{A_d}. \tag{1.6}$$

Let p, q, r, respectively, be d-functions rel. to $C \times C$, $C \times L_2$, $L_2 \times L_2$

with $p = p^*$ and $r = r^*$ where $p^*(s, t) = p(t, s)$, $r^*(s, t) = r(t, s)$. For fixed s and t it has been shown in MT (2), paragraph 7, that

$$|p(s, \cdot)|_{C_d} \leqq h(C, C, p); \qquad |p(\cdot, t)|_{C_d} \leqq h(C, C, p); \quad |q(\cdot, t)|_{C_d} \leqq h(C, L_2, q) \tag{1.7}$$

$$|q(s, \cdot)|_{L_{2d}} \leqq h(C, L_2, q); \qquad |r(s, \cdot)|_{L_{2d}} \leqq h(L_2, L_2, r); \quad |r(\cdot, t)|_{L_{2d}} \leqq h(L_2, L_2, r). \tag{1.8}$$

Given $u \in C$ and $y \in L_2$ the L-S-integrals

$$\varphi_u(s) = \int_0^1 u(t) d_t p(s, t) \qquad \psi_y(s) = \int_0^1 y(t) d_t q(s, t) \tag{1.9}$$

$$f_u(s) = \int_0^1 u(t) d_t q(t, s) \qquad h_y(s) = \int_0^1 y(t) d_t r(s, t) \tag{1.10}$$

define functionals φ_u and ψ_y in C_d and functionals f_u and h_y in L_{2d}. See MT (2), paragraph 9.

Objective: Let $u(s)$ and $v(s)$ map $[0, 1]$ into R_1 and be indefinite integrals of functionals in L_2. We say that u and v are in IL_2. *Necessary and sufficient conditions that the bilinear sum*

$$S(u, v;\ p, q, r) = F(u, v, p) + F(u, \dot{v}, q) + F(v, \dot{u}, q) + F(\dot{u}, \dot{v}, r) = 0$$

for every u, v in IL_2 will be obtained. It is seen that $S(u, v;\ p, q, r) = S(v, u;\ p, q, r)$. The bilinear sum S includes the classical second variation as a very special case. Its theory will be developed so as to extend Hilbert space theory and the theory of integro-differential equations. The present note contains preliminary theorems of fundamental nature.

2. *Lemma 2.1 and Formula (2.5); Measurability.*—We need the relation

$$h(C, C, k) \leqq h(A, B, k) \qquad [A = C \text{ or } L_2,\ B = C \text{ or } L_2]. \tag{2.1}$$

It follows from the definition of $h(A, B, k)$ and the relations $|\eta|_A \leqq |\eta|_C$, $|\zeta|_B \leqq |\zeta|_C$ that

$$|\eta_r \zeta_n \Delta_{rn}(k)| \leqq |\eta|_A |\zeta|_B h(A, B, k) \leqq |\eta|_C |\zeta|_C h(A, B, k). \tag{2.2}$$

On taking the sup. of the extreme members of (2.2) with $|\eta|_C \leqq 1$, $|\zeta|_C \leqq 1$ (2.1) follows. Thus $h(C, C, k) < \infty$ for a d-function k, and we shall show that k is then measurable. That k is measurable over the complement of the open square $(0, 1) \times (0, 1)$ is immediate, since $k(\cdot, 1)$ and $k(1, \cdot)$ are of B.V. by (1.7), and (1.8). According to Theorem 2.1 of MT (1)

$$\lim_{(s, t) \to (0, 0)} k(a - s, b - t) = k(a, b) \qquad (s \geqq 0, t \geqq 0) \tag{2.3}$$

at each point (a, b) of $[0, 1) \times [0, 1)$. Let $[0, 1) \times [0, 1)$ be divided into a network of squares Q_{rm}

$$\frac{r-1}{n} \leqq s < \frac{r}{n} \quad \frac{m-1}{n} \leqq t < \frac{m}{n} \qquad (r, m = 1, \ldots, n).$$

Let k_n map $[0, 1) \times [0, 1)$ into R_1, with $k_n(s, t)$ constant on Q_{rm} and $k(s, t) = k_n(s, t)$ when $s = \dfrac{r-1}{n}, t = \dfrac{m-1}{n}$. It is trivial that k_n is measurable. It follows from (2.3) that $k_n(s, t) \to k(s, t)$ over $[0, 1) \times [0, 1)$ as $n \to \infty$, so that k is measurable.

LEMMA 2.1. *With q defined as above the integral*

$$\mu(s, t) = \int_0^s q(a, t)da \qquad [(s, t) \in E' \times E''] \tag{2.4}$$

properly extended over the (s, t)-plane defines a d-function μ rel. to $L_2 \times L_2$ with $h(L_2, L_2, \mu) \leqq h(C, L_2, q)$.

Since $q(\cdot, t)$ is of B.V. for fixed t by (1.7), the integral (2.4) exists. For fixed s, $q(s, \cdot)$ is absolutely continuous over E'' by (1.8), since its Hellinger integral is finite. Hence $\mu(s, t)$ is continuous over E'' for fixed s. It is obvious that $\mu(\cdot, t)$ is continuous over E' for fixed t. Moreover $\mu(0, t) = \mu(s, 0) = 0$ for $t \in E''$ and $s \in E'$. So defined over $E' \times E''$, μ can be extended in a unique manner over R_2 so as to be canonical rel. $E' \times E''$. See MT (2), Lemma 6.1. It remains to show that $h(L_2, L_2, \mu) \leqq h(C, L_2, q)$. To that end let η and ζ be s-functions belonging to E' and E'', respectively, with $|\eta|_{L_2} \leqq 1$, $|\zeta|_{L_2} \leqq 1$. For fixed $s \in E''$ set $\zeta_n \Delta_n[q(s, \cdot)] = \lambda(s)$, $\zeta_n \Delta_n[\mu(s, \cdot)] = \tau(s)$. Then τ is a. c., and $\dot{\tau}(s) = \lambda(s)$ p.p. (presque partout) over E'. Moreover λ is canonical rel. E', since $q(\cdot, t)$ is so canonical for fixed t. It follows from Theorem 1.2 that $|\lambda(s)| \leqq |q(s, \cdot)|_{L_{2t}}$ and so from (1.8) that $|\lambda(s)| \leqq h(C, L_2, q)$. Then

$$|\eta_r \zeta_n \Delta_{rn}(\mu)| = |\eta_r \Delta_r(\tau)| \leqq \left|\int_0^1 \eta(s)\lambda(s)\, ds\right| \leqq h(C, L_2, q)$$

using the Schwarz lemma. The concluding relation of the lemma follows.

Given a measurable d-function k set

$$k_1(s, t) = \limsup_{n \to \infty} [k\left(s + \frac{1}{n}, t\right) - k(s, t)]n \qquad (n = 1, 2, \ldots).$$

Then k_1 is measurable and equals the s-derivative of k where it exists. One defines k_2 similarly with reference to t. Since p, q, r have been shown to be measurable, p_1, p_2, etc., are measurable. We shall establish the basic formula.

$$\int_0^s da \int^1 \dot{u}(t)d_t q(a, t) = \int^1 \dot{u}(t)d_t \int_0^s q(a, t)da = \int_0^1 \dot{u}(t)d_t\mu(s, t) \qquad [u \in IL_2]. \tag{2.5}$$

Note first that

$$\int_0^1 |q_2(s, t)|dt \leqq |q_2(s, \cdot)|_{L_2} = |q(s, \cdot)|_{L_{2t}} \leqq h(C, L_2, q) \tag{2.6}$$

using a known property of Hellinger integrals and (1.8). Since $q(s, \cdot)$ is a. c. and (2.6) holds

$$\mu(s, t) = \int_0^s d\beta \int_0^t q_2(\beta, \alpha)d\alpha = \int_0^t d\alpha \int_0^s q_2(\beta, \alpha)d\beta \tag{2.7}$$

using Tonelli's Theorem. [See Fubini, Tonelli.] The left member of (2.5) becomes

$$\int_0^s da \int_0^1 \dot{u}(t)q_2(a, t)dt = \int_0^1 \dot{u}(t)[\int_0^s q_2(a, t)da]dt = \int_0^1 \dot{u}(t)\mu_2(s, t)dt$$

using Tonelli's Theorem and (2.7). Relation (2.5) follows on noting that $\mu(s, \cdot)$ is a. c. over [0, 1] for fixed s.

3. Special d-Functions.—Certain special d-functions will be needed to decompose S. A *triangle* d-function k shall be such that $k = k^*$ and $k(s, t) = k(s, 1)$ for $0 \leqq s \leqq t \leqq 1$. For an *edge* d-function $k(s, t) = 0$ for $s < 1$; as canonical rel. $E' \times E''$, k is determined by the values of $k(1, t)$ for $0 \leqq t \leqq 1$. For a *vertex* d-function $k(s, t) = 0$ on $E' \times E''$ except at most at $(1, 1)$; k is then determined by $k(1, 1)$. Each such function is required to be canonical rel. $E' \times E''$. Given p, q, r as previously set

$$r_p(s, t) = \int_0^s \int_0^t p(\alpha, \beta)d\alpha\, d\beta \qquad [(s, t) \in E' \times E'']$$
$$r^{(q)}(s, t) = \int_0^s q(a, t)da + \int_0^t q(a, s)da.$$

Let τ_p be the triangle function with $\tau_p(s, 1) = p(s, 1)$ on E'.
Let $\tau^{(p)}$ be the triangle function with $\tau^{(p)}(s, 1) = \int_0^s p(a, 1)da$ on E'.
Let $e^{(q)}$ be the edge function with $e^{(q)}(1, t) = q(1, t)$ on E''.
Let ν_p be the vertex function with $\nu_p(1, 1) = p(1, 1)$.
We shall establish the relation

$$\int_0^1 u(s)d_s\int_0^1 v(t)d_t\tau_p(s, t) = u(1)v(1)p(1, 1) - \int_0^1 p(s, 1)d[u(s)v(s)] \quad (3.1)$$

for u, v in IL_2. On integrating by parts, see Saks, p. 102.

$$\int^1 v(t)d_t\tau_p(s, t) = \int_0^s v(t)dp(1, t) = v(s)p(1, s) - \int_0^s \dot{v}(t)p(1, t)dt$$
$$\int_0^1 u(s)d_s \int_0^1 v(t)d_t\tau_p(s, t) = \int_0^1 u(s)d[v(s)p(1, s)] -$$
$$\int_0^1 u(s)\dot{v}(s)\, p\, (1, s)ds = u(1)v(1)p(1, 1) - \int_0^1 \dot{u}(s)v(s)p(1, s)ds -$$
$$\int_0^1 u(s)\dot{v}(s)p(1, s)ds \quad (3.2)$$

from which (3.1) follows.

We note the relations, valid for $u, v \in IL_2$

$$\int_0^1 u(s)d_s\int_0^1 v(t)d_t\tau_p(s, t) = \int_0^1 u(s)v(s)dp(s, 1) \quad (3.3)$$
$$\int_0^1 u(s)d_s\int_0^1 \dot{v}(t)d_t\tau^{(p)}(s, t) = \int_0^1 u(s)\dot{v}(s)p(s, 1)\, ds \quad (3.4)$$
$$\int_0^1 u(s)d_s\int_0^1 \dot{v}(t)d_te^{(q)}(s, t) = u(1)\int_0^1 \dot{v}(t)dq(1, t) \quad (3.5)$$
$$\int_0^1 u(s)d_s\int_0^1 v(t)d_t\nu_p(s, t) = u(1)v(1)p(1, 1) \quad (3.6)$$

first verifying the existence of the integrals involved.

Relation (3.3) follows from (3.1) on integration by parts, and (3.4)–(3.6) are immediate.

LEMMA 3.1. *The functionals r_p and $r^{(q)}$ are d-functions rel. $L_2 \times L_2$,*

$\tau^{(p)}$ and $e^{(q)}$ are d-functions rel. $C \times L_2$, while τ_p and v_p are d-functions rel. $C \times C$.

That $r^{(q)}$ is a d-function rel. $L_2 \times L_2$ follows from Lemma 2.1 on noting that $r^{(q)} = \mu + \mu^*$. Continuing

$$\eta_r \zeta_n \Delta_{rn}(r_p) = \int_0^1 \int_0^1 \eta(s)\zeta(t)p(s, t)\, ds\, dt$$

from which it follows that

$$h(L_2, L_2, r_p) \leqq \text{max. } |p(s, t)| \leqq h(C, C, p).$$

Given a d-function, $k(s, t)$ and a partition π'' of E'' we denote by $\Delta_n^{(2)}(k)$ the functional with values $\Delta_n^{(2)}[k(s, \cdot)] = k(s, t_n) - k(s, t_{n-1})$. If η is an s-function and $g \in A_d$ then $\int_0^1 \eta(s)dg(s) = \eta_r \Delta_r g$. Then for $k(s, \cdot) \in B_d$ and ζ an s-function

$$\eta_r \zeta_n \Delta_{rn}(k) = \eta_r \Delta_r[\zeta_n \Delta_n^{(2)}(k)] = \eta_r \Delta_r \int_0^1 \zeta(t) d_t k(s, t). \qquad (3.7)$$

In finding the sup. of the sum (3.7) we can suppose $\pi' = \pi''$ without loss of generality. With this supposed, using (3.7)

$$|\eta_r \zeta_n \Delta_{rn}(\tau_p)| = |\eta_r \Delta_r \int_0^s \zeta(t) d_t p(1, t)| = |\eta_r \zeta_r \Delta_r[p(1, \cdot)]| = |\int_0^1 \eta(s)\zeta(s)\, dp(1, s)|$$

$$|\eta_r \zeta_n \Delta_{rn}(\tau^{(p)})| = |\int_0^1 \eta(s)\zeta(s)p(1, s)ds|$$

$$|\eta_r \zeta_n \Delta_{rn}(e^{(q)})| = |\eta(1) \int_0^1 \zeta(t)dq(1, t)|$$

from which it follows that

$$h(C, C, \tau_p) \leqq T[p(1, \cdot)] \leqq h(C, C, p) \qquad (3.3)'$$

$$h(C, L_2, \tau^{(p)}) \leqq |p(1, \cdot)|_{L_2} \leqq h(C, C, p) \qquad (3.4)'$$

$$h(C, L_2, e^{(q)}) \leqq |q_2(1, \cdot)|_{L_2} \leqq h(C, L_2, q). \qquad (3.5)'$$

Finally $h(C, C, v_p) \leqq |p(1, 1)| \leqq h(C, C, p)$ and the proof of the lemma is complete.

The fact that the special functions r_p, $\tau^{(p)}$, etc., are d-functions permits us to draw upon the properties (1.7) to (1.10) and to invert the order of integration in (3.3) to (3.6).

4. A Model Form for S.—Three formulae are needed.

$$F(u, v, \tau_p) = F(u, v, v_p) - F(u, \dot{v}, \tau^{(p)}) - F(v, \dot{u}, \tau^{(p)}) \qquad (4.1)$$

$$F(u, v, p) = F(\dot{u}, \dot{v}, r_p) \qquad [\text{if } p(s, 1) = 0 \text{ on } E'] \qquad (4.2)$$

$$F(\dot{u}, \dot{v}, r^{(q)}) = -F(u, \dot{v}, q) - F(v, \dot{u}, q) \qquad [\text{if } q(1, t) = 0 \text{ on } E'']. \qquad (4.3)$$

Formula (4.1) is a consequence of (3.1), using (3:6) and (3.4), while (4.2)

results on performing both integrations in $F(u, v, p)$ by parts. To verify (4.3) perform the first integration in $F(u, \dot{v}, q)$ by parts. Then

$$F(u, \dot{v}, q) = -\int_0^1 \dot{u}(s)\left[\int_0^1 \dot{v}(t)d_t q(s, t)\right] ds = \quad (4.4)$$
$$-\int_0^1 \dot{u}(s)d_s \int_0^s da \int_0^1 \dot{v}(t)\, d_t q(a, t)$$

$$= -\int_0^1 \dot{u}(s)d_s \int_0^1 \dot{v}(t)d_t\mu(s, t) \qquad \text{[using (2.5)]} \quad (4.5)$$

$$F(v, \dot{u}, q) = -\int_0^1 \dot{v}(s)d_s \int_0^1 \dot{u}(t)d_t\mu(s, t) \qquad \text{(from (4.5))}$$

$$= -\int_0^1 \dot{u}(s)d_s\int_0^1 \dot{v}(t)d_t\mu(t, s) \quad (4.6)$$

since μ is a d-function rel. to $L_2 \times L_2$. Formula (4.3) is a consequence of (4.5) and (4.6) and the definitions of μ and $r^{(q)}$.

DEFINITION. *Two sets of admissible d-functions* (p, q, r) *and* (p', q', r') *are termed equivalent if* $S(u, v;\ p, q, r) = S(u, v;\ p', q', r')$ *for every* (u, v) *in* IL_2.

This equivalence relation is transitive, reflexive and symmetric. Two equivalences $(p, q, r) \sim (p', q', r')$ and $(p'', q'', r'') \sim (p''', q''', r''')$ imply the equivalence $(p + p'', q + q'', r + r'') \sim (p' + p''', q' + q''', r' + r''')$. Moreover $(p, q, r) \sim (p', q', r')$ implies $(cP, cq, cr) \sim (cp', cq', cr')$ for every constant c.

THEOREM 4.1. *The d-functions* (p, q, r) *are equivalent to a unique set* $(\bar{p}, \bar{q}, \bar{r})$ *of d-functions in which* $\bar{q}$ *is an edge d-function and* $\bar{p}$ *a vertex d-function.*

Since $p - \tau_p$ vanishes on the edges of $E' \times E''$ (4.2) implies that $(p - \tau_p, 0, 0) \sim (0, 0, r_p - r_{\tau_p})$ while (4.1) implies that $(\tau_p, 0, 0) \sim (\nu_p, -\tau^{(p)}, 0)$. On adding the trivial equivalence $(0, q, r) \sim (0, q, r)$ to these equivalences $(p, q, r) \sim (\nu_p, q', r')$ for admissible q' and r'. There thus exist d-functions $(p', q', r') \sim (p, q, r)$ with p' a vertex d-function. From (4.3)

$$(0, q' - e_{q'}, 0) \sim (0, 0, -r^{(q')} + r^{(eq')})$$

since $q' - e_{q'}$ vanishes on E''. It follows that $(p', q', r') \sim (p', e_{q'}, \bar{r})$ for an admissible $\bar{r}$, thereby establishing the existence of a *model* set $(\bar{p}, \bar{q}, \bar{r})$ of the required type.

It follows from (3.5) and (3.6) that

$$S(u, v;\ p, q, r) = F(\dot{u}, \dot{v}, \bar{r}) + u(1)v(1)\bar{p}\,(1,1) + u(1)\int_0^1 \dot{v}(t)d_t\bar{q}(1, t) +$$
$$v(1)\int_0^1 \dot{u}(t)d_t\bar{q}(1, t). \quad (4.7)$$

To show that the *model* set $(\bar{p}, \bar{q}, \bar{r})$ is uniquely determined by S let i represent a functional in IL_2 with $i(s) \equiv 1$.

Let $u_a (0 \leqq a < 1)$ be a functional in IL_2 such that $u(1) = 0$ and u_a is p.p. the characteristic function of $[0, a]$. Then $u_a(s) = s - a$ for $0 \leqq s \leqq a < 1$ while $u_a(s) = 0$ for $a \leqq s \leqq 1$. It follows from (4.7) that

$$\bar{p}(1, 1) = S(i, i;\ p, q, r) \quad (4.8)'$$

$$\bar{q}(1, a) = S(i, u_a;\ p, q, r) \qquad (0 \leqq a < 1) \qquad (4.8)''$$

$$\bar{r}(a, b) = S(u_a, u_b;\ p, q, r) \qquad (0 \leqq b < 1). \qquad (4.8)'''$$

The remaining values $\bar{q}(1, 1)$, $\bar{r}(a, 1)$ and $\bar{r}(1, b)$ are determined by the values given in (4.8), taking account of the continuity of $\bar{q}(1, \cdot)$ and $\bar{r}(1, \cdot)$. Thus the model set $(\bar{p}, \bar{q}, \bar{r})$ is uniquely determined by S. This completes the proof of the theorem.

In terms of (p, q, r) the corresponding model set $(\bar{p}, \bar{q}, \bar{r})$ may be given as follows for $(a, b) \in E' \times E''$ using (4.8).

$$\bar{p}(1, 1) = p(1, 1) \qquad (4.9)'$$

$$\bar{q}(1, a) = q(1, a) + \int_0^a (t - a)dp(1, t) = q(1, a) - \int_0^a p(1, t)dt \qquad (4.9)''$$

$$\bar{r}(a, b) = r(a, b) - \int_0^a q(s, b)\, ds - \int_0^b q(s, a)\, ds + \int_0^a \int_0^b p(s, t)\, ds\, dt. \qquad (4.9)'''$$

COROLLARY 4.1. *A necessary and sufficient condition that $S(u, v;\ p, q, r) = 0$ for every u, v in IL_2 is that the right members of (4.9) vanish for every (a, b) in $E' \times E''$.*

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*THE COEFFICIENTS OF SCHLICHT FUNCTIONS, IV**

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1. In the preceding note (reference 2) a method was outlined for attacking the coefficient problem of schlicht functions, but lack of space made it necessary to omit certain lines of attack which have proved useful

in investigating the problem. Moreover, during the last two years new results have been obtained.* It is the purpose of this note to give a brief sketch of the present status of the authors' research on the coefficient problem. Although some of the results discussed below antedate note 2, they have not been published elsewhere.

Let S denote the family of all functions

$$w = f(z) = z + a_2z^2 + a_3z^3 + \ldots \tag{1.1}$$

which are regular and schlicht in the circle $|z| < 1$, and let V_n denote the region in euclidean space of $2n - 2$ real dimensions which is composed of all points $(a_2, a_3, \ldots, a_n)$ belonging to functions f of S. The region V_n is closed, and it is topologically equivalent to the closed $(2n - 2)$-dimensional full sphere. The coefficient problem is the problem of determining V_n for $n = 2, 3, \ldots$.

It was pointed out in reference 2 that any function $w = f(z)$ belonging to a boundary point of V_n satisfies a differential equation of the form

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 P(w) = Q(z), \tag{1.2}$$

where

$$P(w) = \sum_{\nu=1}^{n-1} \frac{A_\nu}{w^\nu} \qquad Q(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu}{z^\nu}, \qquad B_{-\nu} = \overline{B}_\nu. \tag{1.3}$$

The function $Q(z)$ is non-negative on $|z| = 1$ and has at least one zero there. There is a one-one correspondence between boundary points of V_n and functions $w = f(z)$ of class S which satisfy a differential equation of the form (1.2). Functions of class S which satisfy a differential equation of the form (1.2) will be called $\mathfrak{D}_n$-functions.

Starting from the finite zeros of $P(w)$ and from $w = \infty$, prolong all possible loci

$$Re \int \sqrt{P(w)}\,\frac{dw}{w} = \text{constant}.$$

The set of these loci will be denoted by Γ_w. It can be shown that Γ_w dissects the plane into finitely many simply-connected domains O_ν in each of which the hyperelliptic integral

$$\zeta = \int \sqrt{P(w)}\,\frac{dw}{w}$$

is regular and defines a schlicht mapping of the domain onto a half-plane or a strip. The structure of Γ_w formed the underlying basis of the method and results described in reference 2.

2. Let $2m$ arcs ($m \geq 1$) of the unit circumference $|z| = 1$ be identified in pairs such that: (i) the circle $|z| \leq 1$ with boundary arcs identified in this way is a closed surface of genus zero (therefore topologically equivalent to the sphere); (ii) if two identified arcs abut, the common end-point is a zero of Q with at most one exception; (iii) in the identification of any pair of arcs the metric defined by

$$|dZ|^2 = \left|\frac{Q(z)}{z^2}\right| |dz|^2$$

is preserved. Two points which are identified with each other may be said to be equivalent. There are the following classes of equivalent points: an interior point of $|z| < 1$ which is equivalent only to itself; an interior point of an arc on $|z| = 1$ which is equivalent to exactly one other point; an end-point of an arc (vertex) which may be equivalent to several other vertices on $|z| = 1$. Let $z_1, z_2, \ldots, z_k$, $k \geq 1$, be a class of equivalent points on $|z| = 1$, and let $2\sigma_\nu$, $\sigma_\nu \geq 0$, be the order of the zero of $Q(z)$ at z_ν. Write

$$\sigma = \sum_{\nu=1}^{k} (\sigma_\nu + 1),$$

and let $N_1, N_2, \ldots, N_k$, be half-neighborhoods interior to $|z| < 1$ at the points $z_1, z_2, \ldots, z_k$, respectively. In N_ν let

$$Z_\nu(z) = \int_{z_\nu}^{z} \sqrt{Q(t)}\,\frac{dt}{t} \tag{2.1}$$

and define

$$\tau = \{Z_\nu(z)\}^{2/\sigma}, \qquad z \subset N_\nu. \tag{2.2}$$

If the branches of the functions (2.1) are suitably chosen, the images of the half-neighborhoods N_ν will fit together in the τ-plane and will form a complete neighborhood of $\tau = 0$. In this manner we define local uniformizers at each point of the closed surface which becomes, therefore, a Riemann surface of genus zero.

Of the classes of equivalent points on $|z| = 1$ we distinguish one and denote it by Σ. We remark that if $\sigma = 1$ for a class of equivalent points $z_1, z_2, \ldots, z_k$ on $|z| = 1$, then $k = 1$. Moreover, by (ii) there is at most one point on $|z| = 1$ for which $\sigma = 1$. We suppose that Σ satisfies the following condition: (iv) Σ may be any class of identified points on $|z| = 1$ subject only to the restriction that if $\sigma = 1$ for some point on $|z| = 1$, then Σ is that point. Two functions Q will be considered equal if their ratio is a positive constant. Then Q plus an identification of arcs on $|z| = 1$ and a choice of Σ satisfying (i), (ii), (iii) and (iv) define a point

in a space S_n. Each point of S_n defines a Riemann surface having a distinguished point corresponding to Σ. We have the following result:

To each point $\mathbf{p}$ *of* S_n *there belongs a unique* $\mathfrak{D}_n$*-function* $w = f(z)$ *which maps the closed Riemann surface belonging to* $\mathbf{p}$ *onto the closed* w*-plane and carries* Σ *into* $w = \infty$. *It satisfies the differential equation*

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 P(w) = Q(z), \tag{2.3}$$

where Q *is given by the point of* S_n *and*

$$P(w) = \frac{A_1}{w} + \frac{A_2}{w^2} + \ldots + \frac{A_{n-1}}{w^{n-1}}.$$

To each $\mathfrak{D}_n$*-function there corresponds a point of* S_n *and, if* $f(z)$ *is not algebraic, it belongs to a unique point of* S_n.

We say that the function $w = f(z)$ realizes the identification of arcs on $|z| = 1$ satisfying the conditions (i), (ii) and (iii). The given identification may also be realized in a stepwise fashion or continuously. For example, given an identification satisfying (i), (ii) and (iii) and a distinguished class or cycle Σ, let $I_1', I_1'', I_2', I_2'', \ldots, I_m', I_m''$ be the pairs of identified arcs on $|z| = 1$. By the definition of Σ it is clear that there is always one pair of abutting identified arcs, say I_1', I_1'', with a common end-point $e^{-i\alpha}$ not in Σ. It then follows that $Q(z)$ must be zero at $z = e^{-i\alpha}$. Let I' and I'' be identified subarcs of I_1' and I_1'' abutting at $e^{-i\alpha}$. Then there is a function

$$v(z, u) = u\{z + b_2z^2 + b_3z^3 + \ldots\}, \; 0 < u < 1, \tag{2.4}$$

which identifies I' and I''. The function $v(z, u)$ maps $|z| < 1$ onto $|v| < 1$ minus a piecewise analytic arc whose two edges correspond to I' and I'', respectively. If $m > 1$, the arcs I' and I'' may be allowed to increase continuously until they coincide with I_1' and I_1''. This defines a function $v(z, u)$ in the interval $u_1 \leq u \leq 1$, $u_1 > 0$. The function $v_1 = v(z, u_1)$ maps I_1' and I_1'' into the opposite edges of a slit interior to the unit circle and it maps $I_2', I_2'', \ldots, I_m', I_m''$ into a new set of identified arcs $J_2', J_2'', \ldots, J_m', J_m''$ lying on the unit circumference in the plane of v_1. The identification of arcs on $|v_1| = 1$ satisfies the same postulates, and we may repeat the process. The following theorem is true:

Let $w = f(z)$ *be a* $\mathfrak{D}_n$*-function belonging to a given point of* S_n *and satisfying*

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 P(w) = Q(z). \tag{2.5}$$

Then there is a function

$$v(z, u) = u\{z + b_2(u)z^2 + b_3(u)z^3 + \ldots\}, \qquad 0 < u \leq 1,$$

which is regular and schlicht in $|z| < 1$, maps $|z| < 1$ onto $|v| < 1$ minus a finite set of analytic arcs, and satisfies

$$\left(\frac{z}{v}\frac{\partial v}{\partial z}\right)^2 Q(v, u) = Q(z), \tag{2.6}$$

where

$$Q(v, u) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu(u)}{v^\nu}, \qquad B_{-\nu}(u) = \overline{B_\nu(u)},$$

is non-negative on $|v| = 1$ with at least one zero there. Moreover, $v(z, u)$ is continuous with respect to u and, except at finitely many points, is differentiable and satisfies

$$u\frac{\partial v}{\partial u} = v\frac{1 + e^{i\alpha}v}{1 - e^{i\alpha}v}, \tag{2.7}$$

where $\alpha = \alpha(u)$ is a piecewise continuous function of u and $Q(e^{-i\alpha(u)}, u) = 0$. The terminal conditions are

$$v(z, 1) = z, \qquad \lim_{u \to 0} \frac{v(z, u)}{u} = f(z),$$

$$Q(z, 1) = Q(z), \qquad \lim_{u \to 0} Q(uw, u) = P(w).$$

We remark that (2.7) is the differential equation derived by Löwner for functions mapping $|z| < 1$ onto $|v| < 1$ minus any Jordan arc not passing through the origin. However, the equation (2.6) for the special class of functions under consideration is an automatic consequence of the continuous realization of the identification.

As u varies from 1 to 0, the point $(b_2(u), b_3(u), \ldots, b_n(u)$ traces a curve which extends from the origin $b_2 = 0, b_3 = 0, \ldots, b_n = 0$ and tends to the boundary point of V_n defined by $w = f(z)$ as u tends to 0. This curve lies entirely in the interior of V_n except for its end-point at $u = 0$. Curves such as this have been used in a numerical calculation concerning the boundary of V_4 which has been carried out since the winter of 1946–1947 under a project sponsored by the Office of Naval Research. The computation is not yet complete, and we shall therefore give only a brief sketch of it.

One purpose of the computation is to prove that $|a_4| \leq 4$ for functions $f(z)$ of class S. This part of the program consists of three parts: (*a*) proof that points of the boundary of V_4 corresponding to functions of the form

$$f(z) = \frac{z}{(1 - e^{i\theta}z)^2} = z + 2^{i\theta}z^2 + 3e^{2i\theta}z^3 + 4e^{3i\theta}z^4 + \ldots \tag{2.8}$$

have the local maximum property with respect to neighboring points; (*b*) proof that over a large part of the boundary $|a_4| \leq 4 - \delta$, where δ is some positive quantity; (*c*) calculation of points on the remainder of the boundary by integrating a system of differential equations obtained from (2.6) and (2.7).

Equating like powers of z in the equation (2.7) we obtain differential equations for the coefficients $b_\nu(u)$ of the function $v(z, u)$. By integration we then obtain Löwner's integral formulas for the coefficients $b_\nu(u)$. From the formula for $b_4(0) = a_4$ it can be shown that if there is a constant c such that $|\alpha(u) - c| \leq 0.08697$, $0 \leq u \leq 1$, then $|a_4| \leq 4$ with equality if and only if $\alpha(u)$ is almost everywhere equal to a constant. This establishes the desired local maximum property of the functions (2.8).

If $f(z)$ is a function of class S which maximizes $Re(a_4)$, then it satisfies an equation (1.2) where the coefficients A_ν, B_ν in (1.3) are of the form

$$\begin{cases} A_1 = a_2^2 + 2a_3, & A_2 = 3a_2, & A_3 = 1 \\ B_0 = 3a_4, & B_1 = 3a_3, & B_2 = 2a_2, & B_3 = 1. \end{cases}$$

Moreover, $Q(z)$ is of the form

$$Q(z) = \frac{1}{z^3}(z + e^{-i(\varphi + \psi)})^2(z^2 - 2pze^{i\varphi} + e^{2i\varphi})(z^2 - 2qze^{i\psi} + e^{2i\psi}),$$

where $p \geq 1$, $q \geq 1$. Comparing the like powers of z we find that

$$a_2 = e^{i(\varphi + \psi)} - qe^{-i\psi} - pe^{-i\varphi}$$

$$3a_3 = e^{2i(\varphi + \psi)} - 4pe^{i\varphi} - 4qe^{i\psi} + e^{-2i\varphi} + e^{-2i\psi} + 4pqe^{-i(\varphi + \psi)}.$$

We add the condition that $|a_2| \leq 2$ and substitute these expressions in the Prawitz form of the area-principle (see reference 1). It can be shown that if γ_1 and γ_2 are numbers satisfying $1 < \gamma_1 \leq 1.1$, $1.25 \leq \gamma_2 < \infty$, then the only possible maxima of a_4 in the region where $\min(p, q) \leq \gamma_1$ or $\max(p, q) \geq \gamma_2$ at which $a_4 > 3.6$ occur for $\min(p, q) = \gamma_1$ or $\max(p, q) = \gamma_2$. Hence, to exclude the region $\min(p, q) \leq \gamma_1$, $\max(p, q) \geq \gamma_2$ from consideration it is sufficient to show that $a_4 < 4$ when $p = \gamma_1$, $\gamma_1 \leq q \leq \gamma_2$ and when $p = \gamma_2$, $\gamma_1 \leq q \leq \gamma_2$. Suppose then that

$$\gamma_1 \leq p \leq q \leq \gamma_2. \tag{2.9}$$

From considerations of symmetry we may also suppose that $0° \leq \varphi \leq 60°$.

The final step in the calculation is to compute points on the boundary of V_4 corresponding to points of S_4 satisfying (2.9) (and also $0° \leq \varphi \leq 60°$). If $re^{i\varphi}$, $\rho e^{i\psi}$ are the roots in $|v| < 1$ of the function $Q(v, u)$ in (2.6), we obtain from (2.7) the four equations

$$\begin{cases} u\dfrac{dr}{du} = r\dfrac{1-r^2}{1-2r\cos(\alpha+\varphi)+r^2}, & u\dfrac{d\rho}{du} = \rho\dfrac{1-\rho^2}{1-2\rho\cos(\alpha+\psi)+\rho^2}, \\ u\dfrac{d\varphi}{du} = 2r\dfrac{\sin(\alpha+\varphi)}{1-2r\cos(\alpha+\varphi)+r^2}, & u\dfrac{d\psi}{du} = 2\rho\dfrac{\sin(\alpha+\psi)}{1-2\rho\cos(\alpha+\psi)+\rho^2}. \end{cases}$$

We observe that $\alpha(u) = \varphi + \psi - \pi$. If the starting values satisfy (2.9) (as we suppose), then it can be shown that

$$\left|\frac{d\alpha}{du}\right| \leq \frac{2}{\gamma_1 - 1}. \tag{2.10}$$

We begin at $u = 1$ with values corresponding to the range (2.9) and whenever $\alpha(u)$ deviates from a constant by an amount not exceeding 0.08697, we know that $|a_4| \leq 4$. Since values of $\alpha(u)$ will be obtained at a discrete set of points, we use (2.10) as a means of estimating intermediate values.

3. We conclude with some results concerning the boundary of the general region V_n. A point on the boundary of V_n at which there is a $(2n-3)$-dimensional hyperplane such that V_n lies entirely on one side of it will be called a supporting point. The set of supporting points will be called the supporting surface and it will be denoted by K_n. It can be shown that K_n is connected and that any function belonging to a point of K_n maps $|z| < 1$ onto the plane minus a single analytic slit without critical points.

Now let M_n be the set of boundary points of V_n whose points belong to functions $f(z)$ which satisfy one and only one equation (1.2) and in this equation the function $Q(z)$ vanishes at only one point of $|z| = 1$ where it has a zero of precise order two. We have the following theorem:

The set M_n is not empty, it is open, and its closure contains the supporting surface K_n. There is a one-one bicontinuous correspondence between points of M_n and a class of Q-functions vanishing at only one point of $|z| = 1$ where there is a zero of precise order two. If **a** *is any point of M_n and $f(z)$ the function belonging to* **a**, *let $Q(z)$ be the corresponding Q-function and let $e^{-i\alpha}$ be the point on $|z| = 1$ where $Q(z)$ vanishes. Let*

$$\psi(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{\beta_\nu}{z_\nu}, \quad \beta_{-\nu} = \bar{\beta}_\nu,$$

have a zero of first or higher order at $z = e^{-i\alpha}$, and let

$$\chi(z) = \frac{1}{2}\int_{e^{-i\alpha}}^{z} \frac{\psi(v)}{\sqrt{Q(v)}} \frac{dv}{v},$$

where $\sqrt{Q(v)}$ is analytic on $|v| = 1$ except for a discontinuity at the point $e^{-i\beta}$ which maps into infinity by $f(z)$. As v passes counterclockwise through $e^{-i\alpha}$, we suppose that $\sqrt{Q(v)}$ passes from negative to positive values. Then the function $f^*(z)$ belonging to any point $\mathbf{a}^*$ of M_n in a sufficiently small neighborhood $U(\mathbf{a}, \epsilon)$ of $\mathbf{a}$ is given by the formula

$$f^*(z) = f(z) - \epsilon \frac{f(z)^2}{2\pi i} \int_{|t|=1} \left(\frac{t f'(t)}{f(t)}\right)^2 \frac{\chi(t)}{\sqrt{Q(t)}\,[f(z) - f(t)]} \frac{dt}{t} + o(\epsilon),$$

where the integration is counterclockwise around $|t| = 1$ beginning and ending at $e^{-i\beta}$. Here the function $\sqrt{Q(t)}$ is to be interpreted as above.

* This paper was written while the authors were engaged on a project sponsored by the Office of Naval Research (Contract N6-ori, 154, Task III).

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[2] Schaeffer, A. C., and Spencer, D. C., "The Coefficients of Schlicht Functions III," PROC. NAT. ACAD. SCI., **32**, 111–116 (1946).

ON ZERMELO'S AND VON NEUMANN'S AXIOMS FOR SET THEORY

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If we assume the theory of truth-functions and the theory of quantifiers from the beginning[1] and disregard the axiom of choice, Zermelo's and von Neumann's systems of set theory may be described as follows.

The membership symbol ϵ connecting variables is taken as primitive in both systems so that any combination of the form $\ulcorner(\alpha \epsilon \beta)\urcorner$ where α and β are any variables is an atomic formula and all further (well-formed) formulae are obtainable from atomic ones by binding quantifiers and applying truth-functional connectives.

In Zermelo's system abstracts (such as $\ulcorner\hat{x}\phi x\urcorner$, $\ulcorner\hat{y}\psi y\urcorner$, etc.), identity and inclusion may be introduced by the following definitions. Let $\ulcorner\phi x\urcorner$ be a formula in which x is free but y is not free, and $\ulcorner\phi y\urcorner$ be the result obtained from $\ulcorner\phi x\urcorner$ by substituting occurrences of y for all free occurrences of x. Let ζ and η each be either a variable which is not x or an abstract.

ZM. $\ulcorner(x \epsilon \hat{y}\phi y)\urcorner$ or $\ulcorner\hat{x}(x \epsilon \phi x)\urcorner$ for $\ulcorner\phi x\urcorner$.

ZI. $\ulcorner(\zeta = \eta)\urcorner$ for $\ulcorner(x)(x \epsilon \zeta \equiv x \epsilon \eta)\urcorner$.

ZS. $\ulcorner(\zeta \subset \eta)\urcorner$ for $\ulcorner(x)(x \epsilon \zeta \supset x \epsilon \eta)\urcorner$.

The axioms of Zermelo's system as revised by Fraenkel and Skolem are the following:[2]

Z1. Axiom of extensionality. $(x = y \supset (x \in z \equiv y \in z))$.
Z2. Axiom of pairing. $(\exists w)(x)(x \in w \equiv (x = y \vee x = z))$.
Z3. Axiom of infinity. $(\exists w)((\exists u)(u \in w) \cdot (x)(x \in w \supset (\exists y)(y \in w \cdot x \subset y \cdot x \neq y)))$.
Z4. Axiom of summation. $(\exists w)(x)(x \in w \equiv (\exists y)(x \in y \cdot y \in z))$.
Z5. Axiom of the set of subsets. $(\exists w)(x)(x \in w \equiv x \subset z)$.
Z6. If $\ulcorner \phi xy \urcorner$ is a formula in which x and y are free but z, w and u are not free, then $\ulcorner (x)(y)(z)((\phi xy \cdot \phi xz) \supset y = z) \supset (\exists w)(x)(x \in w \equiv (\exists y)(y \in u \cdot \phi yx)) \urcorner$ is an axiom.

Von Neumann's system may be described by first introducing new variables a, b, c, d, e, etc., which take as values only classes capable of being members of the other classes (elements). These element variables can be introduced by contextual definitions such as:

NA. $\ulcorner (a)\phi a \urcorner$ for $\ulcorner (x)((\exists y)(x \in y) \supset \phi x) \urcorner$,
NE. $\ulcorner (\exists a)\psi a \urcorner$ for $\ulcorner (\exists x)((\exists y)(x \in y) \cdot \psi x) \urcorner$,

etc., where $\ulcorner \phi a \urcorner$ and $\ulcorner \psi a \urcorner$ are like $\ulcorner \phi x \urcorner$ and $\ulcorner \psi x \urcorner$ except for containing free occurrences of a wherever $\ulcorner \phi x \urcorner$ and $\ulcorner \psi x \urcorner$ contain free occurrences of x. The definitions for abstracts, identity, and inclusion are:

NM. $\ulcorner (a \in \hat{b}\phi b) \urcorner$ or $\ulcorner (a \in \hat{a}\phi a) \urcorner$ for $\ulcorner \phi a \urcorner$.
NI. $\ulcorner (\zeta = \eta) \urcorner$ for $\ulcorner (a)(a \in \zeta \equiv a \in \eta) \urcorner$.
NS. $\ulcorner (\zeta \subset \eta) \urcorner$ for $\ulcorner (a)(a \in \zeta \supset a \in \eta) \urcorner$.

The indispensable axioms of von Neumann's system as revised by Bernays and Goedel are (with non-essential changes) the following:[3]

N1. $x = y \supset (x \in z \equiv y \in z)$.
N2. $(\exists d)(a)(a \in d \equiv (a = v \vee a = c))$.
N3. $(\exists d)((\exists e)(e \in d) \cdot (a)(a \in d \supset (\exists b)(b \in d \cdot a \subset b \cdot a \neq b)))$.
N4. $(\exists d)(a)(a \in d \equiv (\exists b)(a \in b \cdot b \in c))$.
N5. $(\exists d)(a)(a \in d \equiv a \subset c)$.
N6. $(e)(v)((a)(b)(c)((<ab> \in v \cdot <ac> \in v) \supset b = c) \supset (\exists d)(a)(a \in d \equiv (\exists b)(b \in e \cdot <ba> \in v)))$.
N7. If $\ulcorner \phi a \urcorner$ is a formula in which y is not free and all bound variables are element variables, then $\ulcorner (\exists y)(a)(a \in y \equiv \phi a) \urcorner$ is an axiom.

If in the above system we replace N7 by a stronger principle due to Quine:[4] N7′. If $\ulcorner \phi a \urcorner$ is a formula in which y is not free, then $\ulcorner (\exists y)(a)(a \in y \equiv \phi a) \urcorner$ is an axiom, we obtain a system NQ which appears to be somewhat stronger than von Neumann's system.[5] Unlike N7, N7′ is not yet known to be reducible to finitely many axioms. Nevertheless, this system

NQ seems to be an interesting extension of von Neumann's, because in it we may form classes without asking whether the defining predicate is a "constitutive expression,"[6] and the mere replacement of N7 by N7′ does not seem to affect the consistency of the system.[7]

Let Z1′–Z6′, ZM′, ZI′ and ZS′ be the result obtained from Z1–Z6, ZM, ZI and ZS by substituting element variables for all general variables. Then Z2′–Z5′ are just N2–N5, and Z1′, ZM′, ZI′ and ZS′ are special cases of N1, NM, NI and NS by dint of NA. Moreover, by N7, Z6′ is a consequence of N6. In fact, with the help of N7, we can deduce from N6 a principle N6′ somewhat stronger than Z6′: N6′. If $\ulcorner\phi ab\urcorner$ is a formula in which all bound variables are element variables and in which a, b are free but c, d, e are not free, then $\ulcorner(a)(b)(c)((\phi ab \cdot \phi ac) \supset b = c) \supset (\exists d)(a)(a \in d \equiv (\exists b)(b \in e \cdot \phi ba))\urcorner$ is an axiom. Consequently, to every class in Zermelo's theory there corresponds an element in von Neumann's theory, although there may be some element in von Neumann's system to which no class in Zermelo's system corresponds.

Obviously von Neumann's system is stronger than Zermelo's. More precisely, to every theorem on classes in Zermelo's system, there corresponds a theorem on elements in von Neumann's, because, by using definitions such as NA and NE, we can also prove that the result of substituting element variables for all general variables in any theorem of the theory of quantifiers is again a theorem in it. In particular, if there is a contradiction in Zermelo's system, there is also one in von Neumann's. In other words, if von Neumann's system is consistent, so also is Zermelo's.

On the other hand, it is not at all obvious that if Zermelo's system is consistent, so also is von Neumann's. Quine's system of *New Foundations*[8] is not known to be inconsistent, yet his system originally presented in ML, which is related to the former system in a similar manner as von Neumann's is related to Zermelo's, has been shown to be inconsistent.[9] Indeed, as von Neumann has observed,[10] it does seem that none of the known contradictions will arise in his system because, although the troublesome classes can be proved to exist, they are not elements in his system, and contradictions arise only when these classes are taken to be elements. But his remark certainly does not amount to a proof of the relative consistency of his system to Zermelo's. It remains to be demonstrated that his system is consistent if Zermelo's is.

In this paper I only propose to investigate how we can obtain von Neumann's system by a straightforward extension of Zermelo's. The result will bring out more clearly the point that if we regard Zermelo's theory as a functional calculus of the first order, von Neumann's system may be considered to be a functional calculus of the second order.

The extension is effected by first introducing into Zermelo's system a new kind of variables X, Y, Z, U, W, etc., and stipulating that the member-

ship symbol ϵ when connecting a small variable to a large one be also taken as primitive. Accordingly, expressions such as $x \epsilon X$, $x \epsilon Y$, etc., are also atomic formulae, which, together with the other atomic formulae $x \epsilon y$, $x \epsilon z$, etc., may again generate new formulae. Large variables occurring before ϵ are introduced by definitions such as:

ZZM1. $(X \epsilon Y)$ for $(\exists z)(z = X \cdot z \epsilon Y)$,
ZZM2. $(X \epsilon y)$ for $(\exists z)(z = X \cdot z \epsilon y)$.

We further stipulate that the result of replacing all small variables in any theorem of the theory of quantifiers by these large variables is a theorem. The definitions ZM, ZI, ZS, and the principle Z6 are reconstrued as covering also the cases where $\ulcorner \phi x \urcorner$ and $\ulcorner \phi y \urcorner$ in ZM and $\ulcorner \phi x y \urcorner$ in Z6 contain large variables and one or both of ξ and η are large variables or the new abstracts. These reinterpreted definitions and principles will be referred to as ZZM, ZZI, ZZS, ZZ6.

We add one more axiom and one rule of inference to Zermelo's system:[11]

ZZ1. $x = y \supset (x \epsilon X \equiv y \epsilon X)$. $x = Y \supset (x \epsilon z \equiv Y \epsilon z)$. $x = Y \supset (x \epsilon Z \equiv Y \epsilon Z)$. $X = Y \supset (X \epsilon z \equiv Y \epsilon z)$. $X = Y \supset (X \epsilon Z \equiv Y \epsilon Z)$.

ZZ7. If x is free in ϕ and all the bound variables in ϕ are small variables ψ' is like ψ except for containing free occurrences of $\ulcorner \hat{x} \phi \urcorner$ wherever ψ contains free occurrences of X, and ψ is a theorem, then ψ' is also a theorem.

In order to show that the result of thus enlarging Zermelo's system is "equivalent" to von Neumann's, we first correlate the large and small variables with the general and element variables, respectively. Then the last statement of ZZ1 answers to N1, Z2–Z5 answer to N2–N5, and by ZZM, ZZ6 and ZZ7, we can easily deduce ZZ6′ and ZZ7′ which are exact counterparts of N6 and N7. What remains to be shown is that we can prove biconditionals which answer to definitions such as NA and NE. Once we have proved such biconditionals, we can easily see that the two systems are equivalent, for (1) ZZM, ZZI, ZZS are then the exact counterparts of NM, NI, NS, (2) Z1 and the statements in ZZ1 become special cases of the last statement in ZZ1, (3) we can easily prove a principle answering to ZZ7 in von Neumann's system,[12] and (4) the biconditionals answering to ZZM1 and ZZM2 are easily provable by N1 and NE in von Neumann's system.

In order to prove the biconditionals corresponding to definitions such as NA and NE, we observe that the following statements are provable in the enlarged Zermelo system:[13]

T1. If $\ulcorner \phi X \urcorner$ is like $\ulcorner \phi x \urcorner$ except for containing free occurrences of X in place of some free occurrences of x, then $\ulcorner x = X \supset (\phi x \equiv \phi X) \urcorner$ is a theorem.

T2. $(\exists Y)(x \in Y)$. By ZZ1, $x = x$. So, by ZZM, $x \in x(x = x)$, and by ZZ7, $(\exists Y)(x \in Y)$.

T3. $(\exists Y)(X \in Y) \equiv (\exists x)(x = X)$. By ZZM1 and the theory of quantifiers, $(\exists Y)(X \in Y) \equiv (\exists x)(x = X \cdot (\exists Y)(x \in Y))$. Therefore, by T2, we obtain T3.

T4. $(\exists X)(x = X)$. By ZZM, $x = y(y \in x)$. Therefore, by ZZ7, $(\exists X)(x = X)$.

Then we can prove a metatheorem corresponding to NA:

TNA. If $\ulcorner \psi X \urcorner$ is related to $\ulcorner \psi x \urcorner$ as $\ulcorner \phi x \urcorner$ is related to $\ulcorner \phi a \urcorner$ in NA, then $\ulcorner (x)\psi x \equiv (X)((\exists Y)(X \in Y) \supset \psi X) \urcorner$ is a theorem.

Proof: By T1, $x = X \supset (\psi x \equiv \psi X)$. But $(x)\psi x \supset \psi x$. Therefore, $x = X \supset ((x)\psi x \supset ((\exists Y)(X \in Y) \supset \psi X))$. So, $(\exists x)(x = X) \supset ((x)\psi x \supset ((\exists Y)(X \in Y) \supset \psi X))$. Hence, by T3,

$$(x)\psi x \supset (X)((\exists Y)(X \in Y) \supset \psi X). \tag{1}$$

On the other hand, by T1 and T2, $x = X \supset ((X)((\exists Y)(X \in Y) \supset \psi X) \supset \psi x)$. By T4,

$$(X)((\exists Y)(X \in Y) \supset \psi X) \supset (x)\psi x. \tag{2}$$

(1) and (2) taken together give us TNA.

The metatheorem for NE follows by the theory of quantifiers as a corollary.*

[1] For example, as are developed in the first three chapters of Hilbert-Ackermann, *Grundzuege der theoretischen Logik*, Berlin, 1938, or in the first two chapters of Quine, *Mathematical Logic*, New York, 1940. I shall assume Quine's version. Quine's book will be referred to as ML.

[2] See, for example, Ackermann, "Mengentheoretische Begruendung der Logik," *Math. Ann.*, **115**, 1–22. Z6 takes the place of *Aussonderungsaxiom* and *Ersetzungsaxiom*, see footnote 9, page 9 of the cited paper. Compare also consequence 4 of Bernays's axioms IV, Va, Vb, page 3, *J. of Symbolic Logic*, **6**.

[3] See Goedel, *The Consistency of the Continuum Hypothesis*, Princeton, 1940, pages 3–7. The principle N7 takes the place of Goedel's axioms B1–B8 plus an assertion that $(\exists y)(a)(a \in y \equiv a \in a)$; cf. M3 on page 14. $\langle xy \rangle$ is the ordered pair of x and y $(= \{\{x\}\{xy\}\})$.

[4] This is Quine's *202 in ML, page 162.

[5] The difference between N7 and N7′ in their deductive power is not trivial. For example, we can obtain the theory of natural numbers with N1, N2 and N7′ plus the assertion that the empty class is (represented by) an element (cf. *J. of Symbolic Logic*, **13**, p. 129), but the same construction cannot be carried through if we use N7 instead of N7′, because the defining formula for the ancestral relation (ML, p. 216) contains a bound variable which is not an element variable.

[6] See, for example, top of page 9, *J. of Symbolic Logic*, **6**.

[7] Compare middle of page 158, *Ibid.*, **7**.

[8] *Am. Math. Mo.*, **44**, p. 70.

[9] *J. of Symbolic Logic*, **7**, p. 1. The inconsistent system contains N7′ in place of N7. However, whether the replacement of N7′ by N7 would affect the derivation of the Burali-Forti contradiction is not immediately obvious.

[10] Middle of page 228, *J. r. angew. Math.*, **160**.

[11] If we want to obtain N7′ instead of N7, just delete the clause "and all the bound variables in ϕ are small variables" in ZZ7.

[12] Compare the proof of *231, ML, page 171.

[13] T1 can be proved by ZZ1 and ZZ1 in the usual manner. Cf., e.g., ML, page 170, *223.

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THE CULTURE OF PARAMECIUM AURELIA IN THE ABSENCE OF OTHER LIVING ORGANISMS

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The growth requirements of ciliated protozoa are very complex, and little accurate work on this has so far been possible since only a few species have been cultivated in the absence of other living organisms. Sterile cultures of *Tetrahymena geleii*,[1] *Colpoda duodenaria*[2, 3] and *Paramecium multimicronucleatum*[4] have been successfully established. *T. geleii* is thus far the only ciliated protozoan which has been cultured in a synthetic medium in the absence of natural proteins or peptones.[5] In addition to an as yet unidentified "plasmoptyzate factor," presumably of protein nature,[3] *C. duodenaria* requires thiamin, pyridoxin and nicotinic acid in the medium.[6, 7] The growth requirements of *P. micronucleatum* have not yet been established. It has been reported that this organism requires a heat-labile and a heat-stabile factor, both present in yeast press juice and in a bacterial plasmoptyzate.[8]

P. aurelia is routinely grown in a lettuce infusion inoculated with *Aerobacter aerogenes*.[9] While this procedure is satisfactory for general purposes, a standardization of this culture medium would be desirable. This requires the establishment of a culture of *P. aurelia* in a medium devoid of other living organisms, which would sustain the growth of this ciliate at a satisfactory fission rate. After such a medium is developed, a study can be made of the specific growth requirements of this organism.

Sterilization of P. aurelia.—Two different procedures were used to free *P. aurelia* from other living organisms: (*a*) electrical transport of the ciliates against a continuous flow of a sterile agar solution, and (*b*) exposure of *P. aurelia* to penicillin solutions.

(*a*) *Sterilization by Electromigration:* The electromigration tube is shown in figure 1. The tube and the Erlenmeyer containing the wash fluid (NaCl 0.0005 g., KH_2PO_4 0.008 g., $MgCl_2.7\ H_2O$ 0.0035 g., $FeCl_3.6\ H_2O$ 0.006 g., $CaCO_3$ 0.0005 g., agar 1.0 g., lettuce infusion 10 ml., distilled water to make 1 liter) were sterilized separately by autoclaving. Immediately before use the two parts were connected asceptically, and the tube was filled by opening stopcocks *A*, *B* and *C*. The flow of the liquid through the

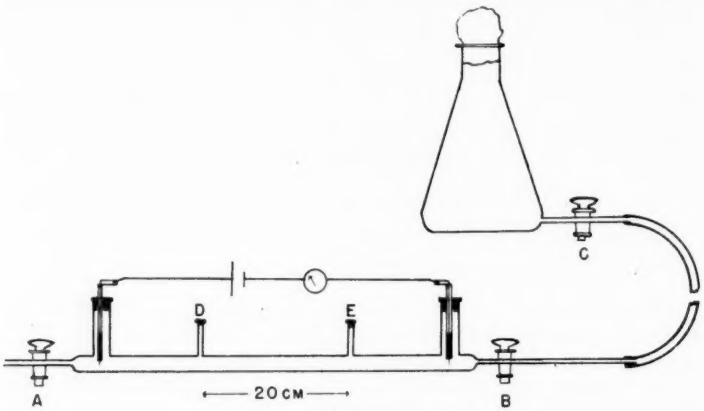

FIGURE 1
Electromigration tube.

tube was adjusted to a rate of 0.6 ml. per minute. The electrodes were connected to a 9-volt battery, thereby producing a current of 0.1 to 0.2 ma. A suspension of *P. aurelia* was injected into side arm *D*. The average migration time of the ciliates from *D* to *E* was about 27 minutes. When the first *P. aurelia* appeared at the side tube *E*, they were removed by means of a sterile syringe and placed in sterile culture fluids of different composition. At the same time samples were distributed to various culture media in order to check for sterility. Subsequent samples of *P. aurelia* can be removed from the tube at point *E* until all of the wash fluid is exhausted. From 30 to 50 ciliates can be recovered from each sample. Under the described conditions about 70 per cent of the samples are sterile.

(*b*) *Sterilization with Penicillin:* Seaman[10] was the first to use penicillin as an agent for the sterilization of protozoans. A wild culture of *C.*

campylum was sterilized by treatment with 5000 units of penicillin per ml. of a 5 per cent proteose peptone broth for 12 hours. In preliminary experiments it was found that *P. aurelia* cannot tolerate concentrations of penicillin higher than 3200 units per ml.

Approximately 20 *P. aurelia* were washed through nine 1-ml. volumes of sterile lettuce infusion, and the small number of organisms which were recovered from the washing process were grown in lettuce infusion inoculated with *A. aerogenes* until the population had reached a concentration of 2000 animals per ml. These paramecia, contaminated by the presence of only a single organism, were then transferred to a solution of 1 per cent yeast autolyzate (Basamin-Busch, kindly furnished by the Anhaeuser Busch Company) saturated with $CaCO_3$, and containing 3200 units of penicillin G-sodium salt per ml. (obtained through the courtesy of Dr. J. M. McGuire of Eli Lilly and Company). This solution had been sterilized by filtration through a sintered glass bacterial filter. After 48 hours in this medium, the organisms were washed through 5 2-ml. volumes of sterile lettuce infusion. The *P. aurelia* from the last wash were then transferred to various media. At the same time samples were distributed to various culture media in order to test for sterility.

(*c*) *Sterility Tests:* *P. aurelia*, treated in either of the two ways described under (*a*) or (*b*), were tested for the presence of contaminating organisms. Aliquots of 0.5 ml. of culture medium to which the organisms had been transferred after the sterilization procedure (taken immediately after the transfer), and aliquots of 0.5 ml. of the culture medium after *P. aurelia* had been in this medium for 5 days were inoculated in the following media: (1) yeast beef agar (Difco), (2) 0.5 per cent yeast extract (Difco), (3) 0.5 per cent yeast extract to which 0.5 per cent sucrose had been added, (4) 0.5 per cent beef extract (Difco), (5) nutrient agar (Difco) to which 0.5 per cent sucrose had been added. The tests were incubated at three different temperatures, 20°, 27°, 37°, under both aerobic and anaerobic conditions. If no contamination occurred, as observed by the absence of cloudiness in the liquid media or absence of colonies on the solid media, in three weeks, the originally isolated *P. aurelia* were considered to be free of any contamination. The organisms were independently tested for sterility by Dr. L. S. McClung of the Department of Bacteriology of Indiana University who confirmed our observations.

Culture in Sterile Media.—Of the various media tried, only the following were able to sustain growth of *P. aurelia:*

(*a*) Equal volumes of 0.5 per cent yeast autolyzate (Basamin-Busch) sterilized by filtration through a sintered glass bacterial filter, and an autoclaved lettuce infusion (20 minutes, 15 pounds), which had been inoculated with *A. aerogenes* 24 hours previous to autoclaving.

(*b*) Equal volumes of 0.5 per cent yeast autolyzate, sterilized by auto-

claving (20 minutes, 15 pounds) and an autoclaved lettuce infusion which had been inoculated with *A. aerogenes* 24 hours previously.

The organisms would multiply in this medium at an average fission rate of 1.7 fissions per day (table 1). The culture has been carried through more than 30 successive transfers. At this time no contamination could be detected by any of the methods mentioned under (*c*). Thus far it has not been possible to make any substitutions for either the particular brand of yeast autolyzate or for the autoclaved bacterial suspension. Various other brands of dried yeast preparations and preparations of fresh yeast were tried but none of these could support the growth of *P. aurelia*. Addition of a bacterial autolyzate in 1 *M* glycine[11] did not allow *P. aurelia* to survive for more than 10 days. No fission took place in this interval. Ultrasonic preparations of an Aerobacter suspension, followed by sterilization by filtration failed to yield preparations which, when added to the yeast autol-

TABLE 1

GROWTH RATE OF *Paramecium aurelia* IN A STERILE MEDIUM

NO. OF 48-HR. TRANSFERS	INITIAL NO. OF *P. aurelia*	NO. OF *P. aurelia* AFTER 48 HRS.	FISSIONS PER DAY
1	81	563	1.4
2	56	371	1.4
3	37	272	1.4
4	85	680	1.5
5	56	681	1.8
10	22	350	2.0
15	51	690	1.8
20	41	524	1.8
25	74	1170	2.0
30	15	216	1.9

yzate, would sustain the growth of the ciliates. When either the lettuce infusion or the autoclaved bacterial culture was omitted, the medium became incapable of supporting the growth of the ciliates; similar results were obtained when the three-component medium (described under (*a*) or (*b*) was filtered through a sintered glass bacterial filter. Rapid death resulted when the autoclaved lettuce–Aerobacter infusion was the only constituent of the medium.

Discussion.—In view of the accumulated knowledge of the nutrition of exacting species of protozoans,[12, 13] it is surprising that *P. aurelia* is capable of growing and multiplying in a medium containing only heat-stabile components. However, the fact that the sterile medium permits only a maximal growth of 1.7 fissions per day, as compared with 4 to 6 fissions per day in a lettuce–Aerobacter infusion, suggests that heat-labile factors may be necessary for optimal growth. The complex nutritional requirements of *P. aurelia* are evident from the fact that a three-component medium is required for sterile growth. It should now be possible to fractionate the

different components into their constituents and thus arrive at a synthetic medium for this organism. All three components of the medium, yeast extract, lettuce infusion and the autoclaved bacterium *A. aerogenes*, contribute essential factors for the nutrition of *P. aurelia*. The observation that this organism will not grow in a medium composed of the three essential components, but filtered through a bacterial filter seems to indicate that some particles of large size are also needed for its growth. However, it might be possible that some essential nutrilite is so firmly bound to the bacterial cell constituents that it had not been liberated in solution by the methods used.

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